Cooperative and axiomatic approaches to the knapsack allocation problem *

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June 11, 2020

Abstract

In the knapsack problem a group of agents wants to fill a knapsack with several goods. Two issues should be considered. The first issue is to decide optimally the goods selected for the knapsack. This issue has been studied in many papers from the literature of Operations Research and Management Science. The second one is to divide the total revenue among the agents. This issue has been studied in few papers, including this one. For each knapsack problem we consider three cooperative games associated. One of them (the pessimistic) was already considered in the literature. The other two (realistic and optimistic) are defined in this paper. The pessimistic and the realistic game have a non-empty core but the core of the optimistic could be empty. Later, we follow the axiomatic approach. We propose two rules. The first one is based on the optimal solution of the knapsack problem. The second one is the Shapley value of the so called optimistic game. We offer axiomatic characterizations of both rules.

Keywords: Knapsack problem; axiomatic; cooperative games.

1 Introduction

A mountaineer is planning a mountain tour with a knapsack, which has a limited size. Thus, he must decide what objects to carry in the backpack. The idea is to select the most important things, given its limited size. This is a classical example of the so called knapsack problem. In general we have a finite set of goods which has to be packed in a knapsack

*Arribillaga acknowledges financial support from the Universidad Nacional de San Luis, through grant 319502, and from the Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), through grant PIP 112-200801-00655. The work of G. Bergantiños is partially supported by research grants ECO2014-52616-R and ECO2017-82241-R from "Ministerio de Economía y Competitividad", and ED431B 2019/34 from "Xunta de Galicia".

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of limited size. Each good j has a profit p_j and a size w_j. We should select a subset of goods whose total size does not exceed the size of the knapsack and whose total profit is a maximum.

The knapsack problem has been applied to various real-world decisions. Examples (see Pisinger and Toth (1998)) include investments (deciding how to split the investment of a fixed amount of money between several business projects) and cargo airlines (deciding how to fill an airplane given the demand of the customers). Other applications (see Bretthauer and Setty (2002)) include financial models, production and inventory management, stratified sampling, the optimal design of queuing network models in manufacturing, computer systems, and health care.

The most popular formulation is the so called 0-1 knapsack problem. There is a finite number of goods (one unit of each good) and it must be decided which ones are selected for the knapsack. The goods can either enter completely (1) or not at all (0). Since the number of goods is finite, there is an optimal solution (the one that maximizes the sum of the profits of the goods included in the knapsack). The first issue addressed is the computational complexity of the optimal solution. Unfortunately, this problem is NP hard (see, for instance, Martello *et al* (2000)). Thus, the optimal solution must be approximated by some algorithms.

There are more general formulations of the knapsack problem. They include the continuous knapsack problem, where fractions of each good can be included; the bounded knapsack problem, where there can be several copies of each good; the d-dimensional knapsack problem, where there are several constraints (for instance weight and volume) on filling the knapsack; the multiple knapsack problem, where there are several knapsacks instead of only one; the multiple choice knapsack problem, where there are several types of object and one object of each type must be chosen; and the non-linear knapsack problem, where the objective function and the constraint are non-linear. Again, the main issue addressed by this literature is how to compute the optimal solution. Pisinger and Toth (1998), Martello *et al* (2000), and Kellerer *et al* (2004) survey this literature.

In all the literature mentioned above it is assumed that there is a single agent involved in the situation. Of course, such agent only cares about what is the optimal solution. Nevertheless in many situations several agents could be involved. As in the classic situation we have a knapsack of limited size which has to be filled with several goods of a given size. But now agents could have different preferences over the importance of the goods. We assume that a group of agents (N) decide which goods (from a set M) should be included in a knapsack of fixed size W. Each good $j \in M$ has a fixed size w_j. The preferences of the agents for the goods are heterogeneous and are modeled by a vector p where for each $i \in N$ and $j \in M$, $p_j^i \in \mathbb{R}_+$ denotes the utility obtained by agent i when one unit of good j is included in the knapsack. We assume that the utility of each agent is linear in the quantities consumed. The goods could be public (if $p_j^i > 0$ for any agent i, then good j could be considered as a public good because every agent benefit from it) or private (if we take $p_j^i > 0$ for agent i and $p_j^k = 0$ when $k \neq i$, then good j could be considered as a private good of agent i).

Now we can define the profit of good j (the p_j of the classical problem) as $\sum_{i \in N} p_j^i$. We also assume (as in the classical model) that agents will select the goods maximizing the total utility (the sum of the utility of all agents). Thus, the computation of the optimal solution (or the approximation obtained) is the first part of the problem. The second part is to divide the cost (or benefits) among the agents. The first part is mainly studied in operations research literature, while the second part is also studied in economics. For instance, in the minimum cost spanning tree problem, Bird (1976), Kar (2002), Dutta and Kar (2004), Tijs *et al* (2006), Bergantiños and Vidal-Puga (2007a), Bogomolnaia and Moulin (2010), and Trudeau (2012) propose several rules for allocating the cost of the optimal solution among the agents. Borm *et al* (2001) give a survey focusing on connection problems, routing (Chinese postman and travelling salesman), scheduling (sequencing, permutation, assignment), production (linear production, flow), and inventory.

As far as we know, the paper by Darmann and Klamler (2014) is the only one in which this second part is studied in the knapsack problem. They focus on the continuous knapsack problem, where the optimal solution could be computed in polynomial time. They consider the following: *"the goal is to divide the cost of the optimally packed knapsack among the individuals in a fair manner. In this paper, we assume that every unit of weight imposes a cost of one, and therefore the total cost of the knapsack is equal to the weight constraint W"*. They then define a family of rules which is characterized by several properties. They also study a particular rule in such family, that divides the cost associated with each good equally among the agents approving that good.

Our paper also considers the second part of the problem, but our approach is different. Darmann and Klamler (2014) consider the case where agents either approve or disapprove each good. Namely, for each i and j, $p_j^i = 1$ when agent i approves good j and $p_j^i = 0$ when agent i disapproves good j. Moreover, our main goal is to divide the total utility generated by the optimal knapsack among the agents.

We first clarify the difference between the two approaches with a trivial example. Consider the knapsack problem with three agents (1, 2, and 3) and two goods (a and b). The size of the knapsack is 1 and the size of each good is also 1. Good a is approved by agents 1 and 2 and good b is approved by agent 3. In our model $p_a^1 = p_a^2 = p_b^3 = 1$ and $p_b^1 = p_b^2 = p_a^3 = 0$. Including good a in the knapsack results in an aggregate utility of 2 (agents 1 and 2 enjoys an utility of 1 and agent 3 enjoys 0). Including good b results in an aggregate utility of 1 (agent 1 and 2 enjoys an utility of 0 and agent 3 enjoys 1). The optimal solution is to include good a in the knapsack. In the rule ϕ^e of Darmann and Klamler (2014) agents 1 and 2 pay 0.5 and agent 3 pays nothing. This means that agent 1 and 2 obtain some earnings (the utility that they get from good a minus the amount that they pay) whereas agent 3 obtains nothing (he receives nothing and pays nothing). In our case agents must decide how to divide the utility generated by the optimal solution (2 in this case) among the agents. Thus, we also consider the possibility that agent 3 is compensated by agents 1 and 2 (because good b is not included) and thus obtains a profit. Actually, one of the allocations that we consider does this.

In this paper we follow a cooperative approach and study how to divide the total utility among the agents. Thus, we implicitly assume that agents who include many of "their goods" in the knapsack could compensate those agents who include few of "their goods" in order to obtain a more fair allocation.

In the literature there is a way of associating a cooperative game with each knapsack problem (see, for instance, Kellerer *et al* (2004)). The value of a coalition S is defined as the utility obtained by agents of that coalition when the knapsack is filled in the worst way for S. We call this game the pessimistic game. It is known that the core of this game is non-empty and contains the allocation induced by the optimal solution. We introduce two alternative cooperative games: the optimistic game and the realistic game. In the optimistic game the value of a coalition S is defined as the utility obtained by the agents of that coalition when the knapsack is filled in the best way for S. It is easy to see that the core of the optimistic game could be empty. In the realistic game the value of a coalition S is defined as the utility obtained by agents of that coalition when agents in $N \backslash S$ fill the knapsack in the best way for $N \backslash S$. We prove that the realistic game has a non-empty core containing the allocation induced by the optimal solution.

We then follow the axiomatic approach: a knapsack rule is a function that for each knapsack problem selects the goods to be included in the knapsack and the way in which the total utility generated by those goods is divided among the agents. We introduce several properties of rules and we discuss some relationships between the properties. One of them is core selection, which says that the allocation should be in the core of the realistic game. In several knapsack problems core selection implies that some agents could receive 0, which seems a little unfair. Thus, we also consider the securement property (inspired by Moreno-Ternero and Villar (2004)), which guarantees all agents a minimum amount. Securement says that each agent must receive at least $(1/n)$ the amount that he obtains when the knapsack is assigned to him. Unfortunately there is no rule that satisfies both properties. Thus we consider two rules: one satisfying each of the properties.

We first consider the rule induced by the optimal solution. This rule allocates to each agent the utility obtained by that agent under the optimal solution. It satisfies core selection but fails securement. We present three characterizations of this rule. In the first one we use core selection and no advantageous splitting. In the second one we use efficiency, maximum aspirations, independence of irrelevant goods, and composition up. In the third one we use efficiency, maximum aspirations, and no advantageous splitting.

We then consider the Shapley value of the optimistic game, which satisfies securement but fails core selection. We characterize it with efficiency and equal contributions.

The rest of the paper is organized as follows. In Section 2 we formally introduce the knapsack problem. In Section 3 we study the three cooperative games associated with the knapsack problem. In Section 4 we introduce the properties, the rules, and the axiomatic characterizations. In Section 5 we present some concluding remarks. In the Appendix we present some omitted proofs of our results. Finally, we give the list of references.

2 The knapsack problem

In the knapsack problem a set of agents want to include some goods in a knapsack of a given size.

We assume that the set of potential agents is infinite. Then, there exists an infinite set $\mathcal{N}$ such that $N \subset \mathcal{N}$.

We focus on the continuous knapsack problem, where it is assumed that goods are perfectly divisible. Then we can select fractions of each good to be included in the knapsack.

A **knapsack problem** is defined as a 5-tuple $P = (N, M, W, w, p)$ where

- $N = \{1, ..., n\}$ denotes a set of agents.
- $M = \{g_1, ..., g_m\}$ denotes the set of goods.
- $W \in \mathbb{R}_+$ is the size of the knapsack.
- $w = \{w_j\}_{j \in M}$ where for each $j \in M$, w_j denotes the size of good j.
- $p = \{p_j^i\}_{i \in N, j \in M}$ where for each $i \in N$ and $j \in M$, $p_j^i \in \mathbb{R}_+$ denotes the utility that agent i obtains for each unit of good j that is included in the knapsack.

Darmann and Klamler (2014) consider the particular case where $p_j^i \in \{0, 1\}$ for each $i \in N$, $j \in M$. Namely, agents approve or disapprove each good.

We introduce some notation used later. Given a knapsack problem P and $M' \subset M$ we denote by $P^{M'}$ the restriction of P to goods in M'. Namely,

$$P^{M'} = \left(N, M', W, \{w_j\}_{j \in M'}, \{p_j^i\}_{i \in N, j \in M'}\right).$$

Given a knapsack problem P and $N' \subset N$ we denote by $P^{N'}$ the restriction of P to agents in N'. Namely,

$$P^{N'} = \left(N', M, W, w, \{p_j^i\}_{i \in N', j \in M}\right).$$

For each $j \in M$,

$$p_j = \sum_{i \in N} p_j^i \tag{1}$$

is a measure of the importance of good j for the set of agents.

For each $S \subset N$ and $j \in M$, $p_j^S = \sum_{i \in S} p_j^i$. Notice that for each $j \in M$, $p_j^N = p_j$.

For each $i \in N$, we denote $p^i = \left(p_j^i\right)_{j \in M}$ the vector of utilities associated with agent i.

The interesting case arises when we can not include in the knapsack all goods, namely, $W < \sum_{j \in M} w_j$. The case $W \geq \sum_{j \in M} w_j$ is solved easily by including all goods in the knapsack. We assume in the rest of the paper that $W < \sum_{j \in M} w_j$.

We say that $x = (x_j)_{j \in M} \in \mathbb{R}^M$ is a **feasible solution** for P if $x_j \in [0,1]$ for each $j \in M$ and $\sum_{j \in M} w_j x_j = W$. We denote by $FS(P)$ the set of feasible solutions for P. As $x_j \in [0,1]$, we assume that one unit of each good is at most admitted. Since $W < \sum_{j \in M} w_j$, $FS(P)$ has many elements.

Each feasible solution x induces a vector of utilities $u(x) = (u_i(x))_{i \in N}$ given by the goods we have included in the knapsack. For each feasible solution x and each $i \in N$,

$$u_i(x) = \sum_{j \in M} p_j^i x_j.$$

We assume that agents choose the goods to be included in the knapsack. They also decide the way in which the total utility generated by the selected goods is divided among them. For any problem P the **set of feasible allocations** is defined as

$$FA(P) = \left\{ (y_i)_{i \in N} \in \mathbb{R}_+^N : \sum_{i \in N} y_i = \sum_{i \in N} u_i(x) \text{ for some } x \in FS(P) \right\}.$$

The first question addressed in the literature (mainly from operations research) is to select the goods to be included in the knapsack in such a way that the aggregated utility of the agents is maximized. Formally,

$$\max_{x \in FS(P)} \sum_{i \in N} u_i(x). \tag{2}$$

In what follows, we assume, without loss of generality, that the goods are sorted in such a way that[1]

$$\frac{p_1}{w_1} \geq ... \geq \frac{p_m}{w_m}.$$

This problem has at least one **optimal solution**. One of them is $x^*(P) = \left\{ x_j^*(P) \right\}_{j \in M}$ defined as

$$x_j^*(P) := \begin{cases} 1 \text{ if } j = 1, ..., s-1 \\ \dfrac{1}{w_s} \left(W - \sum_{k=1}^{s-1} w_k \right) \text{ if } j = s \\ 0 \text{ if } j = s+1, ..., m \end{cases} \tag{3}$$

where s is defined by

$$\sum_{k=1}^{s-1} w_k < W \leq \sum_{k=1}^{s} w_k.$$

[1] This ordering of the goods is not necessarily unique because ties are possible.

When no confusion arises we write x^* instead of $x^*(P)$. We will denote by $X^*(P)$ (or X^*) the set of all optimal solutions in P.

If we assume that $\frac{p_1}{w_1} > ... > \frac{p_m}{w_m}$, we can guarantee that the previous problem has a unique optimal solution.

We denote by $\mathcal{P}$ the class of all knapsack problems and by $\mathcal{P}^*$ the class of knapsack problems where $\frac{p_1}{w_1} > ... > \frac{p_m}{w_m}$.

3 Knapsack cooperative games

In this section we associate with each knapsack problem three cooperative games with transferable utility known as pessimistic, optimistic, and realistic, depending on how the value of a coalition S is defined.

The pessimistic game has been already studied in the literature (see, for instance, Kellerer *et al.* (2004)) and the value of a coalition S is computed in the worst scenario for coalition S. This is the most standard approach and has been used in many different kind of problems. In this case it is assumed that the knapsack is filled including the goods with smaller aggregated utility for agents in S. The optimistic game, inspired in Bergantiños and Vidal-Puga (2007b) and Bergantiños and Lorenzo (2008), is in some sense dual of the pessimistic game because the value of a coalition S is computed in the best scenario for coalition S. Thus, it is assumed that the knapsack is filled including the goods with larger aggregated utility for agents in S. The realistic game tries to be a kind of compromise between the pessimistic and the optimistic game. We take a pessimistic approach in the sense that we allow coalition $N\backslash S$ to fill the knapsack in the best way for them. We take an optimistic approach in the sense that, among all the allocations that give a larger aggregated utility to $N\backslash S$, coalition S can select the one that gives a larger aggregated utility to S.

We study the core of such games. The core of pessimistic and realistic games is always non-empty whereas the core of the optimistic game could be empty.

A **cooperative game with transferable utility** (briefly, a TU game) is a pair (N, v) where $v : 2^N \to \mathbb{R}$ satisfies $v(\varnothing) = 0$. When no confusion arises we write v instead of (N, v).

The **core** of a TU game v is defined as

$$c(v) = \left\{ x \in \mathbb{R}^N : \sum_{i \in N} x_i = v(N) \text{ and for each } S \subset N, \sum_{i \in S} x_i \geq v(S) \right\}.$$

In the pessimistic approach we assume that the knapsack is filled in the worst way for any proper coalition $S \subsetneq N$ and all agents agree to fill the knapsack optimally. Formally, for each knapsack problem P we define the **pessimistic game** v_P^p where,

$$v_P^p(S) = \left\{ \begin{array}{c} \min_{x \in FS(P)} \sum_{i \in S} u_i(x) \text{ if } S \subsetneq N \\ \\ \max\limits_{x \in FS(P)} \sum_{i \in N} u_i(x) \text{ if } S = N \end{array} \right\}.$$

When no confusion arises we write v^p instead of v_P^p.

In the optimistic approach we assume that agents in S can fill the knapsack however they want. Formally, for each knapsack problem P we define the **optimistic game** v_P^o where for each $S \subset N$,

$$v_P^o(S) = \max_{x \in FS(P)} \sum_{i \in S} u_i(x).$$

When no confusion arises we write v^o instead of v_P^o.

In the realistic approach we assume that coalition S chooses its best allocation among those that optimize the space of the knapsack for the coalition $N \backslash S$. Let $X^*(P^{N \backslash S})$ be the set of optimal solutions of the problem $P^{N \backslash S}$. For each knapsack problem P we define the **realistic game** v_P^r where for each $S \subset N$,

$$v_P^r(S) = \max_{x \in X^*(P^{N \backslash S})} \sum_{i \in S} u_i(x).$$

When no confusion arises we write v^r instead of v_P^r.

Remark 1 *It is obvious that for each problem P and each $S \subset N$, $v^p(S) \leq v^r(S) \leq v^o(S)$ and $v^p(N) = v^r(N) = v^o(N)$. Then,*

$$c(v^o) \subset c(v^r) \subset c(v^p).$$

Example 1 *Let P be such that $N = \{1, 2, 3\}$, $M = \{a, b, c\}$, $W = 2$ and $w_j = 1$ for all $j \in M$. Besides the vector p satisfies the following conditions.*

- *Agent 1 is interested in good a but not in the others. Namely, $p_a^1 > 0$ and $p_j^1 = 0$ otherwise.*
- *Agents 2 and 3 prefer b to c and they are not interested in good a. Furthermore, they enjoy good c more than agent 1 enjoys good a. Namely, for each agent $i \neq 1$, $p_b^i > p_c^i > p_a^1$ and $p_a^i = 0$.*
- *Agent 2 is more interested in objects of $M \backslash \{a\}$ than agent 3. Namely, $p_j^2 > p_j^3$, for each $j \in \{b, c\}$.*

We now compute the three games. We detail the computation for coalition $\{2,3\}$*. The worst feasible solution for agents* 2 *and* 3 *is to include goods* a *and* c*. Thus,* $v^p(2,3) = p_c^2 + p_c^3$*. The best feasible solutions for agent 1 are* $\{a\}$, $\{a,b\}$, *and* $\{a,c\}$*. Among them, agents* 2 *and* 3 *prefer* $\{a,b\}$*. Then,* $v^r(2,3) = p_b^2 + p_b^3$*. The best feasible solution for agents* 2 *and* 3 *is to select goods* b *and* c*. Then,* $v^o(2,3) = p_b^2 + p_b^3 + p_c^2 + p_c^3$*.*

T	$v^p(T)$	$v^r(T)$	$v^o(T)$
$\{1\}$	0	0	p_a^1
$\{2\}$	p_c^2	$p_b^2 + p_c^2$	$p_b^2 + p_c^2$
$\{3\}$	p_c^3	$p_b^3 + p_c^3$	$p_b^3 + p_c^3$
$\{1,2\}$	$p_a^1 + p_c^2$	$p_b^2 + p_c^2$	$p_b^2 + p_c^2$
$\{1,3\}$	$p_a^1 + p_c^3$	$p_b^3 + p_c^3$	$p_b^3 + p_c^3$
$\{2,3\}$	$p_c^2 + p_c^3$	$p_b^2 + p_b^3$	$p_b^2 + p_b^3 + p_c^2 + p_c^3$
N	$p_b^2 + p_b^3 + p_c^2 + p_c^3$	$p_b^2 + p_b^3 + p_c^2 + p_c^3$	$p_b^2 + p_b^3 + p_c^2 + p_c^3$

The core of the pessimistic game v^p is non empty and contains $u(x')$ for all $x' \in X^*$ (see, for instance, Kellerer *et al.* (2004)).

The core of the optimistic game v^o could be empty. In Example 1, as $v^o(\{1\}) + v^o(\{2\}) + v^o(\{3\}) > v^o(N)$, then $c(v^o) = \varnothing$. [2]

We now prove that the core of the realistic game v^r is non-empty by showing that $u(x^*)$ belongs to such core.

Theorem 1 *For each knapsack problem* P, $u(x') \in c(v^r)$ *for all* $x' \in X^*$.

Proof. Let P be a problem. Assume, to obtain a contradiction, that there exists $x' \in X^*$ such that $u(x') \notin c(v^r)$. Then, it exists $S \subset N$ such that

$$v^r(S) > \sum_{j \in M} p_j^S x_j'$$

.

Let $x \in X^*(P^{N \setminus S})$ be such that

$$v^r(S) = \sum_{j \in M} p_j^S x_j$$

As $x \in X^*(P^{N \setminus S})$, $\sum\limits_{j \in M} p_j^{N \setminus S} x_j \geq \sum_{j \in M} p_j^{N \setminus S} x_j'$. Then,

$$\begin{aligned} \sum_{i \in N} u_i(x) &= \sum_{i \in N} \sum_{j \in M} p_j^i x_j = \sum_{j \in M} p_j^S x_j + \sum_{j \in M} p_j^{N \setminus S} x_j > \sum_{j \in M} p_j^S x_j' + \sum_{j \in M} p_j^{N \setminus S} x_j' \\ &= \sum_{j \in M} p_j x_j' = \sum_{i \in N} u_i(x'), \end{aligned}$$

[2] As $v^o(\{1\}) + v^o(\{2\}) + v^o(\{3\}) > v^o(N)$, then $c(v^o) = \varnothing$.

which contradicts that $x' \in X^*$. ■

The next example shows that the core of v^r could have other elements besides those that are induced for the optimal solutions (i.e. elements outside of $\{u(x') : x' \in X^*\}$).

Example 2 *Let P be such that $N = \{1, 2, 3\}$, $M = \{a, b, c, d\}$, $W = 5$, $w_a = w_b = w_d = 2$, $w_c = 1$, $p_d^1 = 0.7$, $p_a^1 = p_b^1 = p_c^1 = 0$, $p_a^2 = p_b^2 = 1$, $p_c^2 = p_d^2 = 0$, $p_a^3 = 1, p_b^3 = 0.9$, $p_c^3 = 0.8$, and $p_d^3 = 0$. Then, $p_a = 2$, $p_b = 1.9$, $p_c = 0.8$, $p_d = 0.7$,*

$$\frac{p_a}{w_a} = \frac{2}{2} > \frac{p_b}{w_b} = \frac{1.9}{2} > \frac{p_c}{w_c} = \frac{0.8}{1} > \frac{p_d}{w_d} = \frac{0.7}{2}.$$

The optimal solution is $x^* = (1, 1, 1, 0)$. Namely, we include in the knapsack a, b and c. Now $u(x^*) = (0, 2, 2.7)$.

$v^r(1) = 0$, $v^r(2) = 2$, $v^r(3) = 1.9$, $v^r(1, 2) = 2$, $v^r(1, 3) = 2.7$, $v^r(2, 3) = 3.45$, and $v^r(N) = 4.7$. Then v^r has many core elements different from $u(x^*)$. For instance, $(0.7, 2, 2)$.

4 Knapsack rules and properties

In this section we introduce several properties of rules. We discuss some relationships between the properties. Core selection says that we must select an allocation in the realistic core. Rules selecting allocations in the core could be unfair because agents who want goods which are not in great demand (those with small $\frac{p_j}{w_j}$) could receive zero. Thus, we consider the property of securement, which says that each agent must receive a minimum amount. Unfortunately there is no rule that satisfies both properties.

We then introduce two rules. The first one, based on the optimal solution, satisfies core selection. The second one, based on the Shapley value, satisfies securement. We study the properties satisfied by each rule. We also provide several axiomatic characterizations of both rules.

A rule is a function ϕ assigning to each problem P a pair $\phi(P) = (g(P), f(P))$ where $g(P) \in FS(P)$ and $\sum_{i \in N} f_i(P) = \sum_{i \in N} u_i(g(P))$. Notice that $g(P)$ denote the goods we include in the knapsack and $f(P)$ denotes the way in which the total utility generated by $g(P)$ is divided among the agents.

We now introduce several properties of rules and we discuss some relationships between the properties.

Efficiency says that $f(P)$ is not Pareto dominated in the set of feasible allocations $FA(P)$.

Efficiency (ef)**.** For each problem P, $\sum_{i\in N} f_i(P) = \max_{x\in FS(P)} \sum_{i\in N} u_i(x)$.

In $\mathcal{P}$ efficiency says that $g(P) \in X^*$. In $\mathcal{P}^*$ efficiency means that $g(P) = x^*$.

Symmetry says that if two agents give the same utility to each good, then both receive the same allocation.

Symmetry (sym)**.** For each problem P and each $i, i' \in N$ such that $p^i = p^{i'}$, then $f_i(P) = f_{i'}(P)$.

Monotonicity says that if the valuation of agent i to some goods increases, then the allocation to agent i can not decrease.

Monotonicity (mon)**.** Consider two problems $P = (N, M, W, w, p)$ and $P' = (N, M, W, w, p')$ such that there exists $i \in N$ satisfying $p'^i_j \geq p^i_j$ for all $j \in M$ and $p'^k_j = p^k_j$ for all $j \in M$ and $k \in N\backslash\{i\}$. Then, $f_i(P') \geq f_i(P)$.

Dummy says that if some agent is not interested in any good, then he receives nothing.

Dummy (dum)**.** For each problem P and each $i \in N$ such that $p^i_j = 0$ for each $j \in M$, then $f_i(P) = 0$.

Core selection says that the allocation proposed by the rule should belong to the core of the problem. Because of the definitions, we think that $v^r(S)$ represents better what agents of S could obtain by themselves than $v^p(S)$ or $v^o(S)$. Thus, we select the core of the realistic game for defining this property.

Core selection (cs). For each problem P, $f(P) \in c(v^r)$.

It is clear that core selection implies efficiency, because a feasible allocation, $f(P)$, in the core satisfies that, $\sum_{i\in N} f_i(P) = v^r(N)$.

Assume that we remove a good not selected by the optimal solution, then the allocation proposed by the rule does not change. This property is inspired in the well known principle of independence of irrelevant alternatives (used, for instance, in bargaining problems by Nash (1950)).

Independence of irrelevant goods (iig)**.** Let P be a problem and $j \in M$ satisfying that $x_j = 0$ for any optimal solution x. Then, $\phi(P) = \phi\left(P^{M\backslash\{j\}}\right)$.

Composition up says that we can fill the knapsack in one step or, first fill some part of the knapsack and later the remaining. This property has been used in several economics problems. See for instance the surveys of Thomson (2003, 2015) about bankruptcy problems. Darmann and Klamler (2014) also use this property.

For each problem $P = (N, M, W, w, p)$, $W_1 \leq W$ and $x \in [0,1]^M$ we define the problems

$$\begin{aligned} P(W_1) &= (N, M, W_1, w, p) \text{ and} \\ P(W - W_1, x) &= (N, M_x, W - W_1, w_x, p_x) \end{aligned}$$

where

$$\begin{aligned} M_x &= \{j \in M : x_j < 1\}, \\ (w_x)_j &= w_j (1 - x_j) \text{ for each } j \in M_x, \text{ and} \\ (p_x)^i_j &= p^i_j (1 - x_j) \text{ for each } i \in N \text{ and } j \in M_x \end{aligned}$$

Composition up (cu)**.** For each problem P and each $W_1 \leq W$, if $x = g(P(W_1))$

$$\begin{aligned} g_j(P) &= \begin{cases} g_j(P(W_1)) & j \notin M_x \\ g_j(P(W_1)) + g_j(P(W - W_1, x))(1 - g_j(P(W_1))) & j \in M_x \end{cases} \\ f_i(P) &= f_i(P(W_1)) + f_i(P(W - W_1, x)) \text{ for all } i \in N. \end{aligned}$$

We now introduce two properties closely related. There are actually several papers in which these two properties appear as a single property. No advantageous merging means that no group of agents has incentives to pool their utility and to present themselves as a single agent. No advantageous splitting means that no agent has incentives to divide his utility and to present himself as a group of agents.

Let $P = (N, M, W, w, p)$ and $P' = (N', M, W, w, p')$ be such that $N \subset N'$ and there exists $i \in N$ with $p^i = p'^i + \sum_{k \in N' \setminus N} p'^k$ and $p^k = p'^k$ for all $k \in N \setminus \{i\}$.

No advantageous merging (nam). Then,

$$f_i(P') + \sum_{k \in N' \setminus N} f_k(P') \geq f_i(P).$$

No advantageous splitting (nas). Then,

$$f_i(P') + \sum_{k \in N' \setminus N} f_k(P') \leq f_i(P).$$

Darmann and Klamler (2014) consider the property of pairwise merge-and-split-proofness, which is related in its motivation with nam and nas. Both properties are inspired by the property of strategy-proofness introduced in O'Neill (1982). Actually we define it in the same way as shown in Thomson (2003, 2015). There are two differences between pairwise merge-and-split-proofness and $nam + nas$. First, when one agent is divided into several (or several join together as a single agent), in Darmann and Klamler (2014) each agent must approve different goods. Since our model is more general we allow different agents to approve the same good. Second, in Darmann and Klamler (2014) the property says that agents who do not merge or split should not be affected. In our case (as in the bankruptcy problem) we say that agents that merge or split are not better off.

The idea of the following property is to set an upper bound on the utility received by each agent. In our case, each agent can receive no more than the utility that he receives when he can use the whole knapsack.

For each problem P and each $i \in N$ we define the **maximum aspiration** of agent i as $MA_i(P) = \max_{x \in FS(P)} u_i(x)$. Notice that $MA_i(P) = v^o(i)$.

Maximum aspirations (ma). For each problem P and each $i \in N$, $f_i(P) \leq MA_i(P)$.

The idea of the following property is the dual of the previous one. We try to guarantee each agent a minimum amount. In our case each agent must receive at least $(1/n)$ the utility that he obtains when the knapsack is assigned to him. Following Moreno-Ternero and Villar (2004) we call it securement, as they do for the case of bankruptcy problems.

For each problem P and each $i \in N$ we define the **secure allocation** of agent i as

$$SE_i(P) = \frac{1}{n} \max_{x \in FS(P)} u_i(x).$$

Notice that $SE_i(P) = \frac{v^o(i)}{n}$.

Securement (se). For each problem P and each $i \in N$, $f_i(P) \geq SE_i(P)$.

Equal contributions is a principle widely used in the literature since Myerson (1980) introduced it in TU games. It says that if agent i leaves the problem, the change in the allocation of agent k coincides with the change in the allocation to agent i when agent k leaves the problem.

Equal contributions (ec). For each problem P and each $i, k \in N$,

$$f_i(P) - f_i\left(P^{N\setminus\{k\}}\right) = f_k(P) - f_k\left(P^{N\setminus\{i\}}\right).$$

All the above properties can be considered as desirable for a rule, but clearly there could be incompatibilities between them. For example, if we restrict our attention to rules satisfying core selection (securement) we must leave aside securement (core selection) because the two properties are incompatible. We also prove that under the dummy and efficiency properties, independence of irrelevant goods and securement are incompatible. In the proposition below we study these relationships between the properties.

Proposition 1 (1) *There is no rule satisfying core selection and securement.*

(2) *Let ϕ be a rule satisfying dummy and efficiency. Then, ϕ does not satisfy independence of irrelevant goods or securement.*

Proof. (1) Assume that $\phi = (g, f)$ is a rule that satisfies cs. Consider Example 1. For each $i \in N\setminus\{1\}$, $v^r(i) = \sum_{j \in M} p_j^i x_j^* = u_i(x^*)$ because $x^{*N\setminus\{i\}} = x^* = (0, 1, 1)$. Now,

$$v^r(N) = \sum_{i \in N} u_i(x^*) = \sum_{i \in N\setminus\{1\}} \sum_{j \in M} p_j^i x_j^* = \sum_{i \in N\setminus\{1\}} v^r(i).$$

Then, $c(v^r) = (u_i(x^*))_{i\in N}$. Since ϕ satisfies cs, $f_1(P) = u_1(x^*) = 0$. But $f_1(P) = 0 < SE_1(P) = \frac{p_a^1}{3}$. Then, ϕ does not satisfy se.

(2) Let P be such that $N = \{1, 2\}$, $M = \{a, b\}$, $W = 1$, $w_a = w_b = 1$, $p_a^1 = 1$, $p_b^2 = 0.9$ and $p_b^1 = p_a^2 = 0$. Now $v^o(1) = 1$, $v^o(2) = 0.9$, and $v^o(1, 2) = 1$. Then, $SE_1(P) = 0.5$, $SE_2(P) = 0.45$ and $x^* = (1, 0)$. If ϕ satisfies dum, $f_2\left(P^{\{a\}}\right) = 0$. Now, assume that ϕ satisfies ef, then $g(P) = x^*$. Then, if ϕ satisfies iig, $f_2(P) = f_2\left(P^{\{a\}}\right) = 0$ (since that $g_b(P) = 0$). Then, as $f_2(P) = 0 < 0.45 = SE_2(P)$, ϕ does not satisfy se. ■

Core selection is a quite standard property in the literature. Nevertheless, allocations in the core could be very unfair. In the knapsack problem this could also happen. For instance, in Example 1 there is only one core allocation, which gives 0 to agent 1. Thus, if we try to find a fair allocation sometimes it is better to look outside the core. For instance, in TU games the Shapley value (Shapley, 1953) could be outside the core.

We think that securement is a nice fairness property because it guarantees that all non-dummy agents receive something. For instance, in Example 1 it says that agent 1 receives something.

By Proposition 1 core selection and securement are incompatible. Since we consider both properties to be interesting, we study two rules in the paper: One satisfying core selection and the other satisfying securement.

4.1 The rule induced by the optimal solution

In this section we study a rule that satisfies core selection. We focus on the rule induced by the optimal solution. We fill the knapsack in the optimal way and each agent receives the utility given by the knapsack, i.e. there are no transfers between agents. A general knapsack problem can have several optimal solutions, at this section we restrict our study to $\mathcal{P}^*$, where the optimal solution is unique and then well defined. When we consider rules on $\mathcal{P}^*$, the properties defined in the previous section need to ask the additional requirement that each problem P, P', $P^{M\setminus\{j\}}$, $P(W_1)$, $P(W - W_1, x)$ and $P^{N\setminus\{i\}}$ must be in $\mathcal{P}^*$. We study the properties satisfied by this rule and we give several axiomatic characterizations.

Given $P \in \mathcal{P}^*$, let x^* denote the unique optimal solution of P. Making an abuse of notation we denote the rule induced by x^* also as x^*. Namely, let x^* be the rule defined as $g(P) = x^*$ and $f_i(P) = u_i(x^*)$ for all $i \in N$.

The optimal solution has been used by Darmann and Klamler (2014) for defining a rule. In that paper, the cost associated with each good, selected by the optimal solution, is divided equally among the agents approving such good.

We now study the properties of rule x^*.

Proposition 2 (1) *The rule x^* satisfies efficiency, symmetry, monotonicity, dummy, core selection, independence of irrelevant goods, composition up, no advantageous merging, no advantageous splitting, and maximum aspirations.*

(2) *The rule x^* does not satisfy securement and equal contributions.*

The proof is in Appendix.

In the next theorem we give several axiomatic characterizations of the optimal rule.

Theorem 2 (1) *x^* is the unique rule satisfying core selection and no advantageous splitting.*

(2) *x^* is the unique rule satisfying efficiency, independence of irrelevant goods, composition up, and maximum aspirations.*

(3) *x^* is the unique rule satisfying efficiency, no advantageous splitting, and maximum aspirations.*

Besides, the properties used in the previous characterizations are independent.

The proof is in Appendix.

Remark 2 *If we check the proof of* (1) *in Theorem 2 we realize that we can replace core selection by efficiency and individual rationality (for each problem P, each agent $i \in N$ must receives at least $v_P^r(i)$).*

4.2 The rule induced by the Shapley value

In this section we study a rule satisfying securement. We fill the knapsack in the optimal way and each agent receives the utility given by the Shapley value of the optimistic game associated with the knapsack problem[3]. In this section we consider the set of all problems $\mathcal{P}$. We study the properties satisfied by this rule and we give an axiomatic characterization.

The **Shapley value** of a game v (Shapley, 1953) is denoted by $Sh(v)$. For each $i \in N$ we have that

$$Sh_i(v) = \sum_{S \subset N \setminus \{i\}} \frac{s!\,(n-s-1)!}{n!} \left(v\left(S \cup \{i\}\right) - v\left(S\right)\right).$$

Given $P \in \mathcal{P}$, let x^* denote an optimal solution of P. We define the **optimistic Shapley rule,** denoted by Sh^o, as the rule indued by the Shapley value of the optimistic game. Namely, $Sh^0(P) = (g^o, f^o)$ where $g^o(P) = x^*$ and $f^o(P) = Sh(v_P^o)$.

We now study the properties satisfied by the optimistic Shapley rule.

[3] There are other papers where the it is studied the Shapley value of the optimisitic game. For instance Bergantiños and Vidal-Puga (2007b) study it in minimum cost spanning tree problems.

Proposition 3 (1) *The optimistic Shapley rule satisfies efficiency, symmetry, monotonicity, dummy, maximum aspirations, securement, and equal contributions.*

(2) *The optimistic Shapley rule does not satisfy core selection, independence of irrelevant goods, composition up, no advantageous merging and no advantageous splitting.*

The proof is in Appendix.

We now give a characterization of Sh^o.

Theorem 3 *The optimistic Shapley rule is the unique rule satisfying efficiency and equal contributions.*

Besides, the properties are independent.

The proof is in Appendix.

5 Final Remarks

We summarize the main findings of the paper and we conclude.

In the classical knapsack problem a single agent wants to fill a knapsack with several goods. Thus, this agent has to decide optimally the goods selected for the knapsack. This problem has been studied in many papers of the operations research literature.

We consider the case with several agents with linear and heterogeneous preferences over the goods. Now two issues should be considered. Firstly, as in the single agent case, we select the goods that maximize the aggregated utility of all agents. Secondly, we divide the aggregated utility generated by the selected knapsack among the agents. As far as we know the second issue has been studied in very few papers (including this one).

We assign to each knapsack problem three cooperative games. The pessimistic game has been already studied in the literature. The optimistic and the realistic game have been introduced in this paper. The pessimistic and the realistic have a non-empty core but the optimistic could have an empty core.

We also consider two rules. The first one is based on the optimal solution of the knapsack problem. The second one is the Shapley value of the optimistic game. We offer axiomatic characterizations of both rules. The main advantage of the first one is that it is always in the core of the realistic and the pessimistic games. The main disadvantage is that it could be rather unfair and some agents could get nothing. The rule based on the Shapley value is not so unfair because it guarantees to each agent a minimal utility. The main disadvantage is that it could be outside of the core of the pessimistic game.

Since few papers have been studied this problem there are many things that could be considered. We give a brief list.

In this paper we have studied the Shapley value of the optimistic value. We can also consider the Shapley value of the pessimistic and the realistic game. This is not the objective of this paper but some things can be said. Let P be such that $N = \{1, 2, 3\}$, $M = \{a, b, c\}$,

$W = 1$, $w_a = w_b = w_c = 1$. $p_a^1 = x$, $p_b^1 = p_c^1 = 0$, $p_b^2 = p_b^3 = 1$, $p_a^2 = p_c^2 = p_a^3 = p_c^3 = 0$. Assume that x is small enough. Then, $Sh_1(v^p) = Sh_1(v^r) = \frac{2}{3} > MA_1(P) = x$. Thus, the Shapley value of the realistic game and the Shapley value of the pessimistic game do not satisfy maximum aspirations.

Instead of studying the Shapley value, we can consider the nucleolus of some of the games.

We have associated to each knapsack problem a cooperative game. What happens if we associate a bankruptcy problem. The classical bankruptcy rules produce interesting allocations in this setting?

6 Appendix: Proofs of the results

Proof of Proposition 2. (1) It is obvious that x^* satisfies ef, sym, dum, iig, and ma.

We now prove that x^* satisfies cu. We know that there exists $s \in \mathbb{N}$ such that $x_j^*(P) = 1$ for all $j < s$, $0 < x_s^*(P) \leq 1$, and $x_j^*(P) = 0$ for all $j > s$. Let P and $W_1 \leq W$. Then, it exists $t \leq s$ such that $x_j^*(P(W_1)) = 1$ for all $j < t$, $0 < x_t^*(P(W_1)) \leq 1$, and $x_j^*(P(W_1)) = 0$ for all $j > t$.

Let $x = x^*(P(W_1))$. Assume that $x_t < 1$ and $t < s$ (the other cases are similar and we omit it). Then $M_x = \{t, ..., m\}$, $(w_x)_t = w_t(1 - x_t)$, $(w_x)_j = w_j$ for all $j > t$, $(p_x^i)_t = p_t^i(1 - x_t)$, $(p_x^i)_j = p_j^i$ for each $i \in N$, for all $j > t$. Thus,

$$\begin{aligned} \frac{(p_x)_t}{(w_x)_t} &= \frac{p_t(1-x_t)}{w_t(1-x_t)} = \frac{p_t}{w_t} > \frac{p_{t-1}}{w_{t-1}} = \frac{(p_x)_{t-1}}{(w_x)_{t-1}} > \\ &> \ ... > \frac{p_m}{w_m} = \frac{(p_x)_m}{(w_x)_m}. \end{aligned}$$

It is obvious that, $x_j^*(P) = x_j^*(P(W_1))$ if $j \notin M_x$. Now if $j \in M_x$, we will prove

$$x_j^*(P) = x_j^*(P(W_1)) + x_j^*(P(W - W_1, x))(1 - x_j^*(P(W_1))).$$

We consider several cases.

Case 1: $w_1 + ... + w_j \leq W$. Then $x_j^*(P) = 1$. As, $w_1 + ... + w_t x_t = W_1$, $(w_x)_t + w_{t+1} ... + w_j \leq W - W_1$. Then,

$$x_j^*(P(W - W_1, x)) = 1.$$

Hence,

$$x_j^*(P) = 1 = x_j^*(P(W_1)) + x_j^*(P(W - W_1, x))\ (1 - x_j^{*1}(P(W_1)))$$

Case 2: $w_1 + ... + w_j > W$. Since $j \in M_x$, $w_1 + ... + w_{j-1} < W$. Then, $w_1 + ... + w_j x_j^*(P) = W$ and $w_1 + ... + w_t x_t^*(P(W_1)) = W_1$. We consider two cases.

Case 2.1: $j = t$. Then,

$$W - W_1 = w_j \left(x_j^* (P) - x_j^* (P (W_1)) \right).$$

Since $W - W_1 = (w_x)_j \, x_j^* (P (W - W_1, x))$ and $(w_x)_j = w_j \left(1 - x_j^* (P (W_1))\right)$ we have that

$$\begin{aligned} x_j^* (P (W - W_1, x)) &= \frac{W - W_1}{w_j \left(1 - x_j^* (P (W_1))\right)} \\ &= \frac{w_j \left(x_j^* (P) - x_j^* (P (W_1))\right)}{w_j \left(1 - x_j^* (P (W_1))\right).} \end{aligned}$$

Then,

$$x_j^* (P) = x_j^* (P (W_1)) + x_j^* (P (W - W_1, x)) \left(1 - x_j^* (P (W_1))\right).$$

Case 2.2: $j > t$. Then,

$$(w_x)_t + w_{t+1} ... + x_j^* (P) \, w_j = W - W_1.$$

Hence,

$$x_j^* (P) = x_j^* (P (W - W_1, x)).$$

Now, as $x_j^{*1} (P (W_1)) = 0$,

$$x_j^* (P) = x_j^{*1} (P (W_1)) + x_j^* (P (W - W_1, x)) \ \left(1 - x_j^* (P (W_1))\right)$$

Let f^* be the function f associated with x^*. It is straightforward to see that or all $i \in N$,

$$f_i^* (P) = f_i^* (P (W_1)) + f_i^* (P (W - W_1, x)).$$

Then x^* satisfies *cu*.

By Proposition 1, x^* satisfies core selection.

Let P and P' be as in the definition of *nam* and *nas*. Since $p_j = p'_j$ for all $j \in M$, $x^*(P) = x^*(P')$, and

$$\begin{aligned} f_i^* (P') + \sum_{k \in N' \setminus N} f_k^* (P') &= \sum_{j \in M} p_j'^i f_j^*(P') + \sum_{j \in M} \sum_{k \in N' \setminus N} p_j'^k f_j^*(P') \\ &= \sum_{j \in M} \sum_{k \in N' \setminus N} (p_j'^i + p_j'^k) f_j^*(P') \\ &= \sum_{j \in M} p_j^i f_j^*(P) \\ &= f_i^*(P), \end{aligned}$$

we deduce that x^* satisfies *nam* and *nas*.

(2) Let P be such that $N = \{1, 2\}$, $M = \{a, b\}$, $W = 1$, $w_a = w_b = 1$, $p_a^1 = 1$, $p_b^2 = 0.9$ and $p_b^1 = p_a^2 = 0$. Now $v^o (1) = 1$, $v^o (2) = 0.9$, and $v^o (1, 2) = 1$. Thus, $x^* (P) = (1, 0)$, $f^* (P) = (f_1^*(P), f_1^*(P)) = (1, 0)$, $SE_2 (P) = 0.45$, $x^* \left(P^{\{1\}}\right) = 1$, and $x^* \left(P^{\{2\}}\right) = 0.9$. Thus, x^* does not satisfy *se* and *ec*. ■

Proof of Theorem 2. (1) By Proposition 2, x^* satisfies both properties.

We now prove the uniqueness. Let $\phi = (g, f)$ be a rule satisfying cs and nas.

Given a problem P, we know that there exists $s \in \mathbb{N}$ such that $x_j^*(P) = 1$ for all $j < s$, $0 < x_s^*(P) \leq 1$, and $x_j^*(P) = 0$ for all $j > s$ and

$$\frac{p_1}{w_1} > ... > \frac{p_s}{w_s} > \frac{p_{s+1}}{w_{s+1}} ...$$

Let $i \in N$. We can find $h_i \in \mathbb{N}$ large enough such that

$$\frac{(1-\frac{1}{h_i})p_1^i + \sum\limits_{k \in N:\ k \neq i} p_1^k}{w_1} > ... > \frac{(1-\frac{1}{h_i})p_s^i + \sum\limits_{k \in N:\ k \neq i} p_s^k}{w_s} > \frac{(1-\frac{1}{h_i})p_{s+1}^i + \sum\limits_{k \in N:\ k \neq i} p_{s+1}^k}{w_{s+1}} ... \tag{4}$$

Let $N \subset N'$ be such that $|N' \backslash N| = h_i - 1$. We consider $P' = (N', M, W, w, p')$ such that $p'^i = \frac{p^i}{h_i}$, $p'^k = \frac{p^i}{h_i}$ for all $k \in N' \backslash N$ and $p'^k = p^k$ for all $k \in N \backslash \{i\}$. By nas,

$$f_i(P) \geq f_i(P') + \sum_{k \in N' \backslash N} f_k(P'). \tag{5}$$

Furthermore, by (4),

$$v_{P'}^r(i) = \frac{u_i(x^*(P))}{h_i} \text{ and } v_{P'}^r(k) = \frac{u_i(x^*(P))}{h_i} \text{ for all } k \in N' \backslash N.$$

By cs,

$$f_i(P') \geq \frac{u_1(x^*(P))}{h_i} \text{ and } f_k(P') \geq \frac{u_1(x^*(P))}{h_i} \text{ for all } k \in N' \backslash N.$$

By (5),

$$f_i(P) \geq u_i(x^*(P)). \tag{6}$$

Since x^* satisfies ef and (6),

$$f_i(P) = u_i(x^*(P)) \text{ for all } i \in N.$$

(2) By Proposition 2, x^* satisfies the four properties.

We now prove the uniqueness. Let $\phi = (g, f)$ be a rule satisfying the four properties. Since ϕ satisfies ef, $g(P) = x^*$.

Let s be such that $g_j(P) = 1$ for all $j < s$, $0 < g_s(P) \leq 1$, and $g_j(P) = 0$ for all $j > s$.

We take $W_1 = w_1$. By ef,

$$g_j(P(w_1)) = \begin{cases} 1 & \text{if } j = 1 \\ 0 & \text{otherwise} \end{cases}$$

Let $x = g(P(w_1))$, then $x_1 = 1$, $x_j = 0$ for all $j \in M \backslash \{1\}$ and

$$\begin{aligned} M_x &= \{j \in M : j \geq 2\}, \\ (w_x)_j &= w_j \text{ for each } j \in M_x, \text{ and} \\ (p_x)_j^i &= p_j^i \text{ for each } i \in N \text{ and } j \in M_x \end{aligned}$$

by cu,

$$\begin{aligned} g_j(P) &= \begin{cases} g_j(P(w_1)) & \text{for all } j \notin M_x \\ g_j P(W - w_1, x) & \text{for all } j \in M_x \end{cases}, \\ f_i(P) &= f_i(P(w_1)) + f_i(P(W - w_1, x)) \text{ for all } i \in N. \end{aligned}$$

By iig

$$\phi(P(w_1)) = \phi\left(P(w_1)^{\{1\}}\right).$$

For each $i \in N$, $MA_i\left(P(w_1)^{\{1\}}\right) = p_1^i$. By ma, $f_i\left(P(w_1)^{\{1\}}\right) \leq p_1^i$ for each $i \in N$. By ef, $\sum_{i \in N} f_i\left(P(w_1)^{\{1\}}\right) = \sum_{i \in N} p_1^i$. Thus,

$$f_i(P(w_1)) = f_i\left(P(w_1)^{\{1\}}\right) = p_1^i \text{ for each } i \in N.$$

We now apply cu to problem $P(W - w_1, g(P(w_1)))$ by taking $W_1 = w_2$. Let us make an abuse of notation and denote by $P(w_2)$ the first problem given by cu and by $P(W - w_1 - w_2)$ the second one. Using arguments similar to those used for $P(w_1)$ we can deduce that

$$\begin{aligned} g_j(P(w_2)) &= \begin{cases} 1 & \text{if } j = 2 \\ 0 & \text{otherwise} \end{cases} \\ f_i(P(w_2)) &= p_2^i \text{ for each } i \in N. \end{aligned}$$

If we continue to apply cu we obtain that

$$\begin{aligned} g_j(P) &= \sum_{j=1}^{s-1} g_j(P(w_j)) + g_j\left(P\left(W - \sum_{j=1}^{s-1} w_j\right)\right) \text{ for all } j \in M \text{ and} \\ f_i(P) &= \sum_{j=1}^{s-1} f_i(P(w_j)) + f_i\left(P\left(W - \sum_{j=1}^{s-1} w_j\right)\right) \text{ for all } i \in N, \end{aligned}$$

where for each $j = 1, ..., s-1$,

$$\begin{aligned} g_{j'}(P(w_j)) &= \begin{cases} 1 & \text{if } j' = j \\ 0 & \text{otherwise} \end{cases} \\ f_i(P(w_j)) &= p_j^i \text{ for each } i \in N. \end{aligned}$$

and

$$\begin{aligned} g_j\left(P\left(W-\sum_{j=1}^{s-1} w_j\right)\right) &= \begin{cases} g_s(P) & \text{if } j=s \\ 0 & \text{otherwise} \end{cases} \\ f_i\left(P\left(W-\sum_{j=1}^{s-1} w_j\right)\right) &= p_s^i g_s(P) \text{ for each } i \in N. \end{aligned}$$

Since $g(P) = x^*$, for each $i \in N$,

$$\begin{aligned} f_i(P) &= \sum_{j=1}^{s-1} f_i(P(w_j)) + f_i\left(P\left(W-\sum_{j=1}^{s-1} w_j\right)\right) \\ &= \sum_{j=1}^{s-1} p_j^i + p_s^i g_s(P) = \sum_{j=1}^{s-1} p_j^i x_j^*(P) + p_s^i x_s^*(P) \\ &= u_i(x^*). \end{aligned}$$

(3) By Proposition 2, x^* satisfies the properties.

We now prove the uniqueness by induction on n, the number of agents. Let $\phi = (g, f)$ be a rule satisfying ef, ma and nas.

When $n = 1$, by ef, $g(P) = x^*$ and $f_1(P) = u_1(x^*)$.

We assume that $N = \{1, 2\}$. Given a problem P, let s be as in the definition of the optimal solution x^* given by (3). Since $P \in \mathcal{P}^*$,

$$\frac{p_1^1 + p_1^2}{w_1} > ... > \frac{p_s^1 + p_s^2}{w_s} > \frac{p_{s+1}^1 + p_{s+1}^2}{w_{s+1}} ...$$

Now, let $d_1 \in \mathbb{N}$ be such that

$$\frac{p_1^1 + (1-\frac{1}{d_1})p_1^2}{w_1} > ... > \frac{p_s^1 + (1-\frac{1}{d_1})p_s^2}{w_s} > \frac{p_{s+1}^1 + (1-\frac{1}{d_1})p_{s+1}^2}{w_{s+1}} ... \tag{7}$$

Let $N \subset N'$ be such that $|N' \backslash N| = d_1 - 1$. We consider $P' = (N', M, W, w, p')$ such that $p'^1 = p^1$, $p'^2 = \frac{p^2}{d_1}$ and $p'^k = \frac{p^2}{d_1}$ for all $k \in N' \backslash N$. By nas,

$$f_2(P') + \sum_{k \in N' \backslash N} f_k(P') \leq f_2(P). \tag{8}$$

By ef,

$$f_1(P) \leq f_1(P'). \tag{9}$$

Now, let $P'' = (N, M, W, w, p'')$ such that $p''^1 = p'^1 + \sum_{k \in N' \backslash N} p'^k$ and $p''^2 = p'^2 = \frac{p^2}{d_1}$.

Notice that P' is obtained from P'' when agent 1 in P'' splits in agents $\{1\} \cup (N' \backslash N)$. By nas,

$$f_1(P') + \sum_{k \in N' \backslash N} f_k(P') \leq f_1(P''). \tag{10}$$

By (7),

$$MA_1(P'') = u_1\left(x^*(P'')\right).$$

By ma,

$$\begin{aligned}
f_1\left(P''\right) \leq MA_1(P'') &= u_1\left(x^*(P'')\right) \\
&= \sum_{j\in M} p''^1_j x^*_j(P'') \\
&= \sum_{j\in M} p^1_j x^*_j(P'') + \sum_{j\in M}\sum_{k\in N'\setminus N} p'^k_j x^*_j(P'') \\
&= \sum_{j\in M} p^1_j x^*_j(P') + \sum_{j\in M}\sum_{k\in N'\setminus N} p'^k_j x^*_j(P') \\
&= u_1\left(x^*(P')\right) + \sum_{k\in N'\setminus N} u_k\left(x^*(P')\right).
\end{aligned} \tag{11}$$

By (10) and (11),

$$f_1\left(P'\right) + \sum_{k\in N'\setminus N} f_k\left(P'\right) \leq u_1\left(x^*(P')\right) + \sum_{k\in N'\setminus N} u_k\left(x^*(P')\right). \tag{12}$$

By (12) and ef,

$$f_2\left(P'\right) \geq u_2\left(x^*(P')\right). \tag{13}$$

Similarly, if we take $\bar{k} \in N'\setminus N$ and consider $P''' = (N''', M, W, w, p''')$ such that $N''' = \{1, \bar{k}\}$ and $p'''^1 = p'^1 + p'^2 + \sum_{k\in N'\setminus\left(N\cup\{\bar{k}\}\right)} p'^k$ and $p'''^{\bar{k}} = p'^2 = \frac{p^2}{d_1}$, it can be proved that

$$f_{\bar{k}}\left(P'\right) \geq u_{\bar{k}}\left(x^*(P')\right). \tag{14}$$

Then,

$$f_k\left(P'\right) \geq u_k\left(x^*(P')\right) \text{ for all } k \in N'\setminus N. \tag{15}$$

By (12) and (15),

$$f_1\left(P'\right) \leq u_1\left(x^*(P')\right). \tag{16}$$

By (9) and since $x^*\left(P\right) = x^*\left(P'\right)$ and $p^1 = p'^1$,

$$f_1\left(P\right) \leq u_1\left(x^*(P)\right).$$

Similarly it can be proved that

$$f_2\left(P\right) \leq u_2\left(x^*(P)\right).$$

By ef,

$$f_i\left(P\right) = u_i\left(x^*(P)\right) \text{ for all } i \in N.$$

We now consider the case $n \geq 3$. Assume that the result is true when we have less than n agents and we prove it for n.

We first prove that for any $P \in \mathcal{P}^*$ and any pair of agents $i, k \in N$ $(i \neq k)$

$$f_i\left(P\right) + f_k\left(P\right) \leq u_i\left(x^*(P)\right) + u_k\left(x^*(P)\right). \tag{17}$$

We define $P^{+} = (N^{+}, M, W, w, p^{+})$ such that $N^{+} = N \backslash \{k\}$ and $p^{+i} = p^{i} + p^{k}$ and $p^{+t} = p^{t}$ for all $t \in N^{+} \backslash \{i\}$. By induction hypothesis

$$f_t\left(P^{+}\right) = u_t\left(x^{*}(P^{+})\right) \text{ for all } t \in N^{+}. \tag{18}$$

By nas,

$$f_i(P) + f_k(P) \leq f_i\left(P^{+}\right). \tag{19}$$

By (18) and (19)

$$f_i(P) + f_k(P) \leq u_i\left(x^{*}(P^{+})\right) = u_i\left(x^{*}(P)\right) + u_k\left(x^{*}(P)\right). \tag{20}$$

Fix $i \in N$, by (17)

$$\begin{array}{l}\sum\limits_{k \in N\backslash\{i\}} \left[f_i(P) + f_k(P)\right] \leq \sum\limits_{k \in N\backslash\{i\}} \left[u_i\left(x^{*}(P)\right) + u_k\left(x^{*}(P)\right)\right] \Leftrightarrow \\ (n-1) f_i(P) + \sum\limits_{k \in N\backslash\{i\}} f_k(P) \leq (n-1) u_i\left(x^{*}(P)\right) + \sum\limits_{k \in N\backslash\{i\}} u_k\left(x^{*}(P)\right) \Leftrightarrow \\ (n-2) f_i(P) + \sum\limits_{k \in N} f_k(P) \leq (n-2) u_i\left(x^{*}(P)\right) + \sum\limits_{k \in N} u_k\left(x^{*}(P)\right).\end{array} \tag{21}$$

By ef and since $n \geq 3$,

$$f_i(P) \leq u_i\left(x^{*}(P)\right). \tag{22}$$

Since (22) holds for all $i \in N$ and ef,

$$f_i(P) = u_i\left(x^{*}(P)\right).$$

We now prove that the properties used in the previous characterization are independent.

(1) Let $\bar{P}$ be the problem in Example (2). Let $\phi^{\delta} = \left(g^{\delta}, f^{\delta}\right)$ be such that $g^{\delta}(P) = x^{*}$ for each problem P. Besides, $f^{\delta}(P) = u\left(x^{*}(P)\right)$ if $P \neq \bar{P}$ and $f^{\delta}\left(\bar{P}\right) = (0.7, 2, 2)$. This rule satisfies cs, but fails nas.

Let $\phi^{\gamma} = (g^{\gamma}, f^{\gamma})$ be such that $g^{\gamma}(P) = x^{*}$ for each problem P. Besides, the total utility given by each good j is divided among the agents proportionally to the utility that each agent gives to the goods in x^{*}. Namely, given $i \in N$ and $j \in M$ we define:

$$\begin{aligned} y_j^i &= \frac{\sum\limits_{x_k^{*}>0} p_k^i}{\sum\limits_{i \in N} \sum\limits_{x_k^{*}>0} p_k^i} p_j x_j^{*} \\ f_i^{\gamma}(P) &= \sum_{j \in M} y_j^i. \end{aligned}$$

This rules satisfies nas but fails cs.

(2) Let f^{0} be the rule that selects no good and allocates 0 to each agent. This rule satisfies ma, iig and cu but fails ef.

Let $\phi^{\alpha} = (g^{\alpha}, f^{\alpha})$ be such that $g^{\alpha}(P) = x^*$ for each problem P. Besides, the total utility given by each good is divided equally among the agents given positive utility to such good. Namely, given $i \in N$ and $j \in M$ we define:

$$\begin{aligned} N_j &= \left\{ i \in N : p_j^i > 0 \right\}. \\ y_j^i &= \begin{cases} \frac{1}{|N_j|} p_j x_j^* & \text{if } i \in N_j \\ 0 & \text{otherwise} \end{cases} \\ f_i^{\alpha}(P) &= \sum_{j \in M} y_j^i. \end{aligned}$$

This rule satisfies ef, iig and cu but fails ma.

Let $\phi^{\beta} = \left(g^{\beta}, f^{\beta}\right)$ be such that $g^{\beta}(P) = x^*$ for each problem P. Besides, the total utility is divided as equal as possible among the agents in such a way that no agent gets more than his maximum aspiration. Namely, given a problem P and $i \in N$,

$$f_i^{\beta}(P) = \min\left\{MA_i(P), \alpha\right\} \text{ where } \sum_{i \in N} f_i^{\beta}(P) = \sum_{i \in N} u_i(x^*).$$

This rule satisfies ef, ma, and cu but fails iig.

Let $f^{\pi} = (g^{\pi}, f^{\pi})$ be such that $g^{\pi}(P) = x^*$ for each problem P. Given $i \in N$ and $j \in M$ we define:

$$\begin{aligned} M^{\pi} &= \left\{ j \in M : x_j^* > 0 \right\}, \\ FS^{\pi}(P) &= \left\{ x : \sum_{j \in M} w_j x_j = W \text{ and } x_j = 0 \text{ if } j \notin M^{\pi} \right\} \\ y^i &= \max_{x \in FS^{\pi}(P)} u_i(x) \end{aligned}$$

Now, suppose that $N = \{i_1, ... i_n\}$ such that $y^{i_1} \geq y^{i_2} ... \geq y^{i_n}$. Notice that $u_i(x^*) \leq y_i \leq MA_i(P)$ for all $i \in N$. We define

$$\begin{aligned} f_{i_1}^{\pi}(P) &= \min\{y^{i_1}, \sum_{i \in N} u_i(x^*)\}. \\ f_{i_2}^{\pi}(P) &= \min\{y^{i_2}, \sum_{i \in N} u_i(x^*) - f_{i_1}^{\pi}(P)\}. \\ &\vdots \\ f_{i_h}^{\pi}(P) &= \min\{y^{i_h}, \sum_{i \in N} u_i(x^*) - \sum_{r=1}^{h-1} f_{i_r}^{\pi}(P)\}. \\ &\vdots \\ f_{i_n}^{\pi}(P) &= \sum_{i \in N} u_i(x^*) - \sum_{r=1}^{n-1} f_{i_r}^{\pi}(P). \end{aligned}$$

This rule satisfies ef, ma, and iig but fails cu.
(3) ϕ^0 satisfies ma and nas but fails ef.
ϕ^β satisfies ef and ma but fails nas.
ϕ^γ satisfies ef and nas but fails ma. ■

Proof of Proposition 3. (1) It is obvious that Sh^o satisfies ef.

sym. Assume that agents i and j are symmetric in P. Thus, they are symmetric in the optimistic game v^o. Since the Shapley value satisfies symmetry, both agents receive the same. Thus Sh^o satisfies sym.

mon. Let P, P' and i as in the definition of mon. Since the Shapley value is an average of marginal contributions, it is enough to prove that for each $S \subset N \backslash \{i\}$, we have that

$$v_P^o(S \cup \{i\}) - v_P^o(S) \leq v_{P'}^o(S \cup \{i\}) - v_{P'}^o(S).$$

Since $v_{P'}^o(S) = v_P^o(S)$ it is enough to prove that $v_P^o(S \cup \{i\}) \leq v_{P'}^o(S \cup \{i\})$. Notice that $FS(P) = FS(P')$. Let $y \in FS(P)$ be such that $v_P^o(S \cup \{i\}) = \sum_{k \in S \cup \{i\}} \sum_{j \in M} p_j^k y_j$. Now,

$$\begin{aligned} v_P^o(S \cup \{i\}) &= \sum_{k \in S \cup \{i\}} \sum_{j \in M} p_j^k y_j \leq \sum_{k \in S \cup \{i\}} \sum_{j \in M} p_j'^k y_j \\ &\leq \max_{x \in FS(P')} \sum_{k \in S \cup \{i\}} \sum_{j \in M} p_j'^k x_j = v_{P'}^o(S \cup \{i\}). \end{aligned}$$

dum. Assume that agent i is a dummy in P. Thus, agent i is a dummy in the optimistic game v^o. Since the Shapley value satisfies dummy, agent i receives nothing. Thus Sh^o satisfies dum.

ma. Since the Shapley value is an average of marginal contributions, it is enough to prove that for each problem P, each $i \in N$, and each $S \subset N \backslash \{i\}$ we have that $v_P^o(S \cup \{i\}) - v_P^o(S) \leq MA_i(P)$.

Let $y, y' \in FS(P)$ be such that $v_P^o(S \cup \{i\}) = \sum_{k \in S \cup \{i\}} \sum_{j \in M} p_j^k y_j$ and $v_P^o(S) = \sum_{k \in S} \sum_{j \in M} p_j^k y_j'$. Now,

$$\begin{aligned} v_P^o(S \cup \{i\}) - v_P^o(S) &= \sum_{k \in S \cup \{i\}} \sum_{j \in M} p_j^k y_j - \sum_{k \in S} \sum_{j \in M} p_j^k y_j' \\ &= \sum_{j \in M} p_j^i y_j + \sum_{k \in S} \sum_{j \in M} p_j^k y_j - \sum_{k \in S} \sum_{j \in M} p_j^k y_j' \end{aligned}$$

By definition of y', $\sum_{k \in S} \sum_{j \in M} p_j^k y_j - \sum_{k \in S} \sum_{j \in M} p_j^k y_j' \leq 0$. Then,

$$v_P^o(S \cup \{i\}) - v_P^o(S) \leq \sum_{j \in M} p_j^i y_j \leq \max_{x \in FS(P)} \sum_{j \in M} p_j^i x_j = MA_i(P).$$

se. Let P be a problem and $i \in N$. Since $v^o(i) = SE_i(P)\, n$ and $v^o(S \cup i) \geq v^o(S)$ we have that $Sh_i^{o2}(P) \geq SE_i(P)$.

ec. Let P be a problem and $i,k\in N$. Let (N,v_P^o) be the corresponding optimistic game. Myerson (1980) proved that the Shapley value satisfies equal contributions in TU games. Then,

$$Sh_i\left(N,v_P^o\right)-Sh_i\left(N\backslash\left\{k\right\},v_P^o\right)=Sh_k\left(N,v_P^o\right)-Sh_k\left(N\backslash\left\{i\right\},v_P^o\right).$$

Since $f_i^o\left(P\right)=Sh_i\left(N,v_P^o\right)$ and $f_k^o\left(P\right)=Sh_k\left(N,v_P^o\right)$, it is enough to prove that $f_i^o\left(P^{N\backslash\{k\}}\right)=Sh_i\left(N\backslash\left\{k\right\},v_P^o\right)$ and $f_k^o\left(P^{N\backslash\{i\}}\right)=Sh_k\left(N\backslash\left\{i\right\},v_P^o\right)$. We prove that $f_i^o\left(P^{N\backslash\{k\}}\right)=Sh_i\left(N\backslash\left\{k\right\},v_P^o\right)$ (the other case is similar and we omit it). Since $f_i^o\left(P^{N\backslash\{k\}}\right)=Sh_i\left(N\backslash\left\{k\right\},v_{P^{N\backslash\{k\}}}^o\right)$, it is enough to prove that for each $T\subset N\backslash\left\{k\right\}$, $v_P^o\left(T\right)=v_{P^{N\backslash\{k\}}}^o\left(T\right)$. Notice that,

$$FS\left(P\right)=\left\{x:\sum_{j\in M}w_jx_j\leq W\text{ and }x_j\in[0,1]\forall\ j\in M\right\}=FS\left(P^{N\backslash\{k\}}\right).$$

Then,

$$v_P^o(T)=\max_{x\in FS(P)}\sum_{i\in T}u_i\left(x\right)=\max_{x\in FS\left(P^{N\backslash\{k\}}\right)}\sum_{i\in T}u_i\left(x\right)=v_{P^{N\backslash\{k\}}}^o(S).$$

(2) Since Sh^o satisfies *se* and Proposition 1, we have that Sh^o does not satisfy *cs*.

It is obvious that Sh^o does not satisfy *iig*.

cu. Consider Example 1. Since $v^o\left(1\right)=p_a^1$ and $v^o\left(S\cup 1\right)=v^o\left(S\right)$ when $\varnothing\neq S\subset N\backslash\left\{1\right\}$, we have that $f_1^o\left(P\right)=Sh_1\left(v^o\right)=\frac{1}{3}p_a^1$. We take $W_1=1$. Let $x=f_1^o\left(P\left(W_1\right)\right)$. Since $x_b=1$ and $x_a=x_c=0$

$$\begin{aligned}v_{P(W_1)}^o\left(1\right)&=p_a^1,\\ v_{P(W_1)}^o\left(S\cup\left\{1\right\}\right)&=v_{P(W_1)}^o\left(S\right)\text{ when }\varnothing\neq S\subset N\backslash\left\{1\right\},\end{aligned}$$

Since $x_b=1$ and $x_a=x_c=0$,

$$\begin{aligned}v_{P(W-W_1,x)}^o\left(1\right)&=p_a^1,\\ v_{P(W-W_1,x)}^o\left(S\cup\left\{1\right\}\right)&=v_{P(W-W_1,x)}^o\left(S\right)\text{ when }\varnothing\neq S\subset N\backslash\left\{1\right\},\end{aligned}$$

we have that

$$Sh_1\left(v_{P(W-W_1,x)}^o\right)=Sh_1\left(v_{P(W_1)}^o\right)=\frac{1}{3}p_a^1.$$

Since,

$$f_1^o\left(P\left(W_1\right)\right)+f_1^o\left(P\left(W-W_1,x\right)\right)=\frac{2}{3}p_a^1,$$

we deduce that Sh^o does not satisfies *cu*.

nas. It follows from Theorem 2 (3) and the fact that the Sh^o satisfies *ef* and *ma*.

nam. Let P be such that $N=\{1,2,3\}$, $M=\{a,b,c\}$, $W=1$ and $w_j=1$ for all $j\in M$. Besides the vector p satisfies the following conditions: $p_a^1=\frac{1}{2}$, $p_b^1=0$, $p_c^1=1$, $p_a^2=1$, $p_b^2=1$, $p_c^2=0$, $p_a^3=\frac{3}{4}$, $p_b^3=1$ and $p_c^3=1$. Thus,

T	$v_P^o(T)$
$\{1\}$	1
$\{2\}$	1
$\{3\}$	1
$\{1,2\}$	$\frac{3}{2}$
$\{1,3\}$	2
$\{2,3\}$	2
N	$\frac{9}{4}$

Then, $f^o(P) = Sh(v_P^o) = \left(\frac{2}{3}, \frac{2}{3}, \frac{11}{12}\right)$. Therefore, $f_1^o(P) + f_2^o(P) = \frac{4}{3}$.

Assume that agents 1 and 2 merge in agent 1. Now $N^+ = \{1,3\}$, $p^{+1} = p^1 + p^2$, and $p^{+3} = p^3$. Then,

T	$v_{P^+}^o(T)$
$\{1\}$	$\frac{3}{2}$
$\{3\}$	1
N^+	$\frac{9}{4}$

Then $f^o(P^+) = Sh\left(v_{P^+}^o\right) = \left(\frac{11}{8}, \frac{7}{8}\right)$.Then,

$$f_1^o(P) + f_2^o(P) = \frac{4}{3} < \frac{11}{8} = f_1^o\left(P^+\right),$$

which implies Sh^o does not satisfy nam. ■

Proof of Theorem 3. By Proposition 3 we know that Sh^o satisfies ef and ec.

We now prove the uniqueness. This proof is quite standard in the literature. Let ϕ be a rule satisfying ef and ec. We prove it by induction on n.

When $n = 1$, by ef, $g(P) = x^*$ and $f_1(P) = u_1(x^*)$. Assume that the result is true when we have less than n agents and we prove it for n. By ec, for all $i \in N\backslash\{1\}$,

$$\begin{aligned}
f_i(P) - f_i\left(P^{N\backslash\{1\}}\right) &= f_1(P) - f_1\left(P^{N\backslash\{i\}}\right) \Rightarrow \\
f_i(P) - f_1(P) &= f_i\left(P^{N\backslash\{1\}}\right) - f_1\left(P^{N\backslash\{i\}}\right) \Rightarrow \\
\sum_{i\in N\backslash\{1\}} f_i(P) - (n-1) f_1(P) &= \sum_{i\in N\backslash\{1\}} \left(f_i\left(P^{N\backslash\{1\}}\right) - f_1\left(P^{N\backslash\{i\}}\right)\right) \Rightarrow \\
\sum_{i\in N} f_i(P) - n f_1(P) &= \sum_{i\in N\backslash\{1\}} \left(f_i\left(P^{N\backslash\{1\}}\right) - f_1\left(P^{N\backslash\{i\}}\right)\right).
\end{aligned}$$

Since ϕ satisfies ef, $\sum_{i\in N} f_i(P) = \sum_{i\in N} u_i(x^*)$.

By induction hypothesis, $\sum_{i\in N\backslash\{1\}} \left(f_i\left(P^{N\backslash\{1\}}\right) - f_1\left(P^{N\backslash\{i\}}\right)\right)$ is known. Then,

$$f_1(P) = \frac{\sum_{i\in N} u_i(x^*) - \sum_{i\in N\backslash\{1\}} \left(f_i\left(P^{N\backslash\{1\}}\right) - f_1\left(P^{N\backslash\{i\}}\right)\right)}{n}.$$

Thus, $f_1(P)$ is uniquely determined. Let $i \in N \backslash \{1\}$. By ec,

$$f_i(P) = f_i\left(P^{N\backslash\{1\}}\right) + f_1(P) - f_1\left(P^{N\backslash\{i\}}\right),$$

which means that $f_i(P)$ is uniquely determined.

We now prove that the properties are independent.

ϕ^0, defined as in the proof of Theorem 2, satisfies ec but fails ef.

ϕ^β, defined as in the proof of Theorem 2, satisfies ef but fails ec. ■

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