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ORIGINAL PAPER

Convergence of the approximate cores to the aspiration core in partitioning games

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Abstract The *approximate core* and the *aspiration core* are two non-empty solutions for cooperative games that have emerged in order to give an answer to cooperative games with an empty core. Although the *approximate core* and the *aspiration core* come from two different ideas, we show that both solutions are related in a very interesting way in partitioning games (or superadditive games). In fact, we prove that the *approximate core* converges to the *aspiration core* in partitioning games (or superadditive games).

Keywords Core · Aspiration core · Approximate core · Convergence · Partitioning games

Mathematics Subject Classification 91A12 · 91A40 · 91A44

1 Introduction

The partitioning games have been introduced by Kaneko and Wooders (1982), and recently studied by Solymosi (2008), and Auriol and Marchi (2009), among others. These games are useful in modeling situations with restricted cooperative possibilities between the players, and therefore, only some coalitions may be formed. Certainly, the number of coalitions is exponentially large, and it may not be feasible in practice to consider all of them. It may be the case that some of the players in a coalition may not get to meet or communicate with each other, so that actually only some coalitions may be formed. In other contexts, it could be very hard to form a large coalition and then,

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only small coalitions may play essential roles. But even if all coalitions are allowed, it may still happen that only small coalitions play essential roles, because the game has some special structure, as in the bridge game of Shubik (1971), and the assignment games of Shapley and Shubik (1972).

Partitioning games are represented by a finite set N of players, an a priori set π of coalitions of N (subsets of N) and a payoff function $\bar{v}$ on π. Only coalitions in π play an essential role and players have to be organized through partitions taken from π.[1]

The fundamental concept of a cooperative equilibrium is the core, which always assumes that the grand coalition forms. However, the power of the core concept is limited by the fact that the non-emptiness of the core may be assured only in certain ideal environments. Kaneko and Wooders (1982) give necessary and sufficient conditions on π which guarantee that every partitioning game, associated to (N, π), has non-empty core. These conditions are considered by the authors "extremely restrictive and, without some very special structure on the collection of basic coalitions, we would not expect these conditions to be met". A large and current literature has studied these conditions to provide a graph-theoretical characterization of these families; see, for instance, Aguilera and Escalante (2010).

In this paper, we study and compare two non-empty extensions of the core that give alternative solutions to the restrictive condition established by Kaneko and Wooders (1982). One of the solutions is the *approximate core* which proposes the replication of games to obtain non-empty ϵ-cores if the number of replications is sufficiently large. This idea has been introduced by Wooders (1983)[2] and studied in Kaneko and Wooders (1982), Kovalenkov and Wooders (2003) and Wooders (2008), among others. In this approach, the existence results are based on the fact that, with a finite number of types of players and bounded basic group sizes, large games have non-empty approximate cores.

The other solution concept is the *aspiration core* which proposes that the cooperation (or negotiation) of the players can be supported by overlapping structures of coalitions (not just the grand coalition) called *balanced families*. The *aspiration core* has been introduced by Bennett (1983); (see also, Cross 1967; Albers 1979) and recently, studied by Bejan and Gomez (2012), Cesco (2012) and Arribillaga (2013), among others.

Although the *approximate core* and the *aspiration core* are two solutions that have the same motivation—to give an answer to (partitioning) games with an empty core—they have not yet been compared and linked in the literature. The main contribution of this paper is to show different relations between the *approximate core* and the *aspiration core* in partitioning games. First, we show that the cores of the replicated games, in a subsequence of the replica games, are equal to the *aspiration core* of the (original) game. Second, we prove that the collection of ϵ*-approximate cores* converges to the *aspiration core* when ϵ tends to zero. All the obtained results are completely independent of the set π of feasible coalitions and the payoff functions.

[1] $(N, \pi, \bar{v})$ is called a *game with restricted cooperation* in Pulido and Sánchez (2006). In that paper the grand coalition is always feasible ($N \in \pi$) and the players are not reorganized in partitions taken from π.

[2] The 1981 version of the paper is the Cowles Foundation Discussion Paper No. 612 that was published in 1983.

The paper is organized as follows. In the next section, preliminary definitions and notation are introduced. In sect. 3, *approximate core* and *aspiration core* definitions are presented. In sect. 4, we present the main results.

2 Definitions and notation

A *game with sidepayments* is a pair (N, v), where $N = \{1, \dots, n\}$ is a finite set of players and $v : 2^N \rightarrow \mathbf{R}$ is a characteristic function (with $v(\emptyset) = 0$).[3] The number $v(S)$ is interpreted as the value of the coalition S. In a game with sidepayments, all the coalitions are feasible.

A game with sidepayments (N, v) satisfies the *superadditivity property* if

$$v(S \cup S') \geq v(S) + v(S') \quad \text{for all} \quad S, S' \subset N \quad \text{such that} \quad S \cap S' = \emptyset.$$

A *problem with partial cooperation* is a pair (N, π), where $N = \{1, \dots, n\}$ is a finite set of players and $\pi \subset 2^N$ is a set of coalitions. In a problem with partial cooperation, only coalitions in π, called *basic coalitions*, are feasible. For any non-empty $S \subset N$, we call $p_S = \{T_1, \dots, T_k\}$ a π*-partition* of S iff

$$p_S \subset \pi \text{ and } S = \bigcup_{T \in p_S} T \quad \text{with} \quad T \cap T' = \emptyset \quad \text{for all} \quad T, T' \in p_S \quad \text{such that} \quad T \neq T'.$$

The set of π-partitions of S is denoted by $\mathcal{P}^\pi(S)$.

A game in characteristic function form, (N, v), is called a *partitioning game associated to* (N, π) (Kaneko and Wooders 1982) iff for some real-valued function $\bar{v}$ on π,

$$v(S) = \max_{p_S \in \mathcal{P}^\pi(S)} \sum_{T \in p_S} \bar{v}(T), \quad \text{for all non-empty} \quad S \subset N. \tag{1}$$

Example 1 If N_1 is the set of buyers and N_2 is the set of sellers, (with $N_1 \cap N_2 = \emptyset$), $N = N_1 \cup N_2$ is the set of players and the basic coalitions are the singles or the buyer–seller pairs, *i.e.,* $\pi^* = \{T \subset N : \text{either } |T| = 1 \text{ or } |T| = 2 \text{ and } |T \cap N_i| = 1$ for $i = 1, 2\}$. The partitioning games (N, v) associated to (N, π^*) coincide with the assignment games in Shapley and Shubik (1972). There, they prove that every assignment game has a non-empty core. Note that this proposition is independent of the choice of $\bar{v}$.

For any given N and π, $GS(N, \pi)$ denotes the set of all partitioning games associated to (N, π). Let $GS(N)$ denote the set of all the partitioning games with players in N, *i.e.,* $GS(N) = \cup_{\pi \subset 2^N} GS(N, \pi)$. From now on, we will restrict our attention to games in $GS(N)$, and (N, v) will be always a game in $GS(N)$.

Remark 1 It is easy to check that every game in $GS(N)$ is superadditive. On the other hand, if (N, v) is a superadditive game, it can be checked that (N, v) is a partitioning

[3] As usual, 2^N denotes the set of all the coalitions (subsets) of N.

game associated to (N, π, v), i.e., (N, v) is in $GS(N)$.[4] Therefore, $GS(N)$ is the class of superadditive games.

Possible payoffs of a game (N, v) are described by vectors $x \in \mathbf{R}^n$ that assign a payoff x_i to every $i \in N$. For every $S \subset N$ and $x \in \mathbf{R}^n$, define $x(S) = \sum_{i \in S} x_i$.

One of the most studied and compelling solutions in cooperative games is the *core* introduced by Gillies (1959) and defined by

$$C(N, v) = \{x \in \mathbf{R}^n : x(N) \leq v(N) \quad \text{and} \quad x(S) \geq v(S)\ \forall\ S \subset N\}.$$

Remark 2 The core always assumes the formation of the grand coalition. Then, the set of feasible payoffs for the core is given by $\{x \in \mathbf{R}^n : x(N) \leq v(N)\}$.

3 Non-empty solutions

The power of the core concept is limited by the fact that the non-emptiness of the core cannot be always assured. Kaneko and Wooders (1982) determine necessary and sufficient conditions on π, for every game in $GS(N, \pi)$ to have a non-empty core. Those conditions are extremely restrictive. We will study two (different) non-empty solutions that have emerged in order to give an answer to games with an empty core. One of these solutions is the *approximate core* which proposes the replication of games to obtain non-empty approximate cores if the number of replications is sufficiently large. The other solution is the *aspiration core,* proposing that the cooperation (or negotiation) of the players can be supported by overlapping structures of coalitions (not just the grand coalition) called *balanced families.*

3.1 The approximate core

Given the set of players $N = \{1, \ldots, i, \ldots, n\}$, for each positive integer number r we define $N_r = \{(i, q) : i = 1, \ldots, n \text{ and } q = 1, \ldots, r\}$. The set N_r is called the *set of players of the r-th replication of N*.[5] On the subsets of N_r, we define a function ϱ which associates to each $S \subset N_r$ a vector in $\mathbf{R}^n$ whose i-th entry indicates the number of replications of the player i in S. A subset $S \subset N_r$ is called a basic coalition (of N_r) if and only if $\varrho(S) = \varrho(S')$ for some basic coalition $S' \in \pi$ of the set N. Let π_r be the set of all non-empty basic coalitions of N_r. The idea is that basic coalitions in N_r are copies of the basic coalitions in N.

[4] There could be other pairs (N, π) such that (N, v) is associated to (N, π) for some $\pi \subset 2^N$.

[5] N is identified with N_1. If $S \subset N$, we have that S is identified with $\{(i, 1) : i \in S\}$. Then, we can consider that S is a subset of N_r.

For a given game $(N, v) \in GS(N, \pi)$, the *r-th replica game* (N_r, v_r) generated by (N, v) is defined as the (partitioning) game associated to $(N_r, \pi_r, \bar{v}_r)$ where $\bar{v}_r$ is defined on π_r by:[6]

$$\bar{v}_r(T) = v(T') \quad \text{for all} \quad T \in \pi_r \quad \text{where} \quad T' \in \pi \quad \text{with} \quad \varrho(T) = \varrho(T'). \qquad (2)$$

The next example illustrate the concepts presented previously.

Example 2 Let $N = \{1, 2, 3\}$, and let $\pi^* = \{T \subset N : \text{either } |T| \leq 2\}$, and let $\bar{v} : \pi^* \to \mathbf{R}$ defined by

$$\bar{v}(T) = 2\,|T| - 1 \quad \text{for all} \quad T \in \pi^*.$$

The partitioning game (N, v) associated to $(N, \pi^*, \bar{v})$ is defined by

$$\begin{aligned} &v(N) = 4 \\ &v(\{1, 2\}) = v(\{1, 3\}) = v(\{2, 3\}) = 3 \\ &v(\{i\}) = 1 \quad \text{for all} \quad i \in N. \end{aligned}$$

If N_2 is the *set of players of the* 2*-th replication of* N,

$$N_2 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (3, 2)\}.$$

In the 2*-th replica* π^* is identified with set

$$\{(1, 1)\}, \{(2, 1)\}, \{(3, 1)\}, \{(1, 1), (3, 1)\}, \{(2, 1), (3, 1)\}, \{(1, 1), (2, 1)\}\}$$

and

$$\begin{aligned} &\varrho(\{(i, 1)\}) \text{ is a vector in } \mathbf{R}^4 \text{ whose } i - \text{th entry is 1 and the others are 0, for all } i \in N. \\ &\varrho(\{(1, 1), (3, 1)\}) = (1, 0, 1) \\ &\varrho(\{(2, 1), (3, 1)\}) = (0, 1, 1) \\ &\varrho(\{(1, 1), (2, 1)\}) = (1, 1, 0). \end{aligned}$$

If $T = \{(1, 1), (1, 2), (2, 1), (3, 1)\} \subset N_2$, then $T \notin \pi_2^*$ because $\varrho(T) = (2, 1, 1)$.
If $\hat{T} = \{(1, 1), (1, 2)\} \subset N_2$, then $\hat{T} \notin \pi_2^*$ because $\varrho(\hat{T}) = (2, 0, 0)$.
If $\bar{T} = \{(2, 1), (3, 1)\} \subset N_2$, then $\bar{T} \in \pi_2^*$ because $\varrho(\bar{T}) = (0, 1, 1)$.
In this case,

$$\pi_2^* = \{T \subset N_2 : \text{either } |T| = 1 \text{ or } |T| = 2 \text{ and } |\{j : (i, j) \in T\}| \leq 1 \text{ for } i = 1, 2, 3\}$$

and

$$\bar{v}_2(T) = 2\,|T| - 1 \quad \text{for all} \quad T \in \pi_2^*.$$

The 2*-th replica game* (N_2, v_2) generated by (N, v) is now the partitioning game associated to $(N_2, \pi_2, \bar{v}_2)$.

[6] This replication is due to Kaneko and Wooders (1982).

A payoff x for the game (N_r, v_r) is said to have the *equal-treatment property*, if $x^{iq'} = x^{iq''}$ for each $i \in N$ and for each $q', q'' \in \{1, \ldots, r\}$; *i.e.*, players of the same type are allocated the same amount. If we want to propose as solution for (N, v) some payoffs that emerge from (N_r, v_r), it is necessary that such payoffs satisfy the equal-treatment property. The next lemma ensures that payoffs in the core of a replicated game have the equal-treatment property.

Lemma 1 *Let (N_r, v_r) be the $r-$th replication of the game (N, v). If y is in the core of (N_r, v_r), then y has the equal-treatment property.*

Proof Suppose, by contradiction, that there exists $y \in C(N_r, v_r)$, $i^* \in N$, and j, $j' \in \{1, 2, \ldots, r\}$ such that $y_{(i^*,j)} < y_{(i^*,j')}$. Let p be a π_r-partition of N_r such that $y(N_r) \leq \sum\limits_{T \in p} v_r(T)$. As $y(T) \geq v_r(T)$ for all $T \in p$, $y(T) = v_r(T)$ for all $T \in p$. Let $T_j, T_{j'} \in p$ such that $(i^*, j) \in T_j$ and $(i^*, j') \in T_{j'}$. As p is a π_r-partition, $(i^*, j) \notin T_{j'}$. Let $S = T_{j'} \cup \{(i^*, j)\} \backslash (i^*, j')$, then $S \in \pi_r$ and $v_r(S) = v_r(T_{j'})$. As $y_{(i^*,j)} < y_{(i^*,j')}$, we have that $y(S) < y(T_{j'}) = v_r(T_{j'}) = v_r(S)$ which contradicts that $y \in C(N_r, v_r)$. □

Because of the previous lemma, we can see $C(N_r, v_r)$ as a subset of $\mathbf{R}^n$.

Given $\epsilon > 0$, the *ϵ-core* (Shapley and Shubik 1966) of a game (N, v) is the set

$$C_\epsilon(N, v) = \{x \in \mathbf{R}^n : x(N) \leq v(N) \quad \text{and} \quad x(S) \geq v(S) - \epsilon\, |S| \quad \text{for all} \quad S \subset N\}.$$

The set of payoffs in $C_\epsilon(N_r, v_r)$ that has the equal-treatment property is denoted by $EC_\epsilon(N_r, v_r)$.[7]

Given $\epsilon > 0$, the *ϵ-approximate core* of a game (N, v), denoted by $ApC_\epsilon(N, v)$, is the set of payoffs, with the equal-treatment property, in the ϵ-core of some replica of (N, v) , *i.e.*,

$$ApC_\epsilon(N, v) = \{x \in \mathbf{R}^n : x \in EC_\epsilon(N_r, v_r) \text{ for some integer } r\}.$$

The next proposition follows from Theorem 3.4 in Kaneko and Wooders (1982) and Lemma 1.

Proposition 1 *For all $\epsilon > 0$. The ϵ-approximate core is non-empty for all games (N, v).*

Proof Given $\epsilon > 0$, Theorem 3.4 in Kaneko and Wooders (1982) ensures $C_\epsilon(N_r, v_r) \neq \emptyset$ if r is large enough.[8] The proof of such theorem considers a game $(N, \tilde{v})$ associated to (N, v), and shows that there exists $x \in C(N_r, \tilde{v}_r)$ such that $(x - \epsilon) \in C_\epsilon(N_r, v_r)$ if r is large enough.[9] Since $x \in C(N_r, \tilde{v}_r)$, by Lemma 1, x has

[7] The payoffs in the ϵ-core may not have the equal-treatment property.

[8] On the class of *normalized games* in $GS(N, \pi)$, (games (N, v) such that $v(i) \geq 0$ for all $i \in N$ and $v(N) \leq |N|$) how large r is only depends on π, and it is independent of the function v. In the class of all games $GS(N, \pi)$, how large r is depends on π and v.

[9] As usual, $(x - \epsilon) = (x^{iq} - \epsilon)_{(i,q) \in N \times \{1,\ldots,r\}}$.
The game $(N, \tilde{v})$ is called the *balanced cover* of (N, v).

the equal-treatment property. Therefore, $(x-\epsilon) \in EC(N_r, v_r)$ if r is large enough. Then, $ApC_\epsilon(N, v) \neq \emptyset$. $\square$

3.2 The aspiration core

In order to introduce the *aspiration core* notion, we need some preliminary definitions. A family of coalitions $\sigma \subset 2^N$ *generates* the vector x if $x(S) \leq v(S)$ for all $S \in \sigma$ and $\bigcup_{S\in\sigma} S = N$. The *aspiration core* assumes that feasibility is defined by taking into account payoffs generated by overlapping structures of coalitions called *balanced families.* These are families of coalitions satisfying the following two requirements:

a) Each player has all of his " time" distribuited in the coalitions in which he participates.
b) The amount of " time" that a player contributes to a given coalition is the same for all members of that coalition.

Formally, a family of coalitions $\beta \subseteq 2^N$ is called a *balanced family* if there exists a set of positive real numbers $(\lambda_S)_{S\in\beta}$ satisfying

$$\sum_{\substack{S\in\beta:\\ i\in S}} \lambda_S = 1, \quad \text{for all} \quad i \in N.$$

The numbers $(\lambda_S)_{S\in\beta}$ are the *balancing weights* for β. A balanced family suggests what coalitions should be formed, and its balancing weights are interpreted as the fraction of " time" in which each coalition is active. If $S \in \beta$, then each $i \in S$ devotes λ_S of his " time" to S. Since$\sum_{\substack{S\in\beta:\\ i\in S}} \lambda_S = 1$, each player distributes all his " time" among the coalitions which he belongs to.

Let $\mathcal{B}(N)$ denote the set of all balanced families of N, and let β denote an arbitrary element in $\mathcal{B}(N)$.

Given a game (N, v), define

$$v^*(N) = \max_{\beta\in\mathcal{B}(N)} \sum_{S\in\beta} \lambda_S v(S),$$

where $(\lambda_S)_{S\in\beta}$ is the balancing weight vector of β.

Therefore, $v^*(N)$ is the maximum value that N can obtain if the cooperation (or negotiation) of the players is organized by means of balanced families. Then, the set of *feasible payoffs for the aspiration core* is given by

$$\{x \in \mathbf{R}^n : x(N) \leq v^*(N)\}.$$

Remark 3 If we have a partitioning game (N, v) associated to $(N, \pi, \bar{v})$, it will be natural to work with

$$v^\pi(N) = \max_{\substack{\beta\in\mathcal{B}(N):\\ \beta\subset\pi}} \sum_{S\in\beta} \lambda_T \bar{v}(T)$$

instead of $v^*(N)$. Nevertheless, the two values are equal. Clearly, $v^\pi(N) \leq v^*(N)$. Conversely, let us assume that $v^*(N) = \sum_{S\in\beta^*} \lambda_S v(S)$. For each $S \in \beta^*$, let $p_S \in \mathcal{P}^\pi(S)$ such that $v(S) = \sum_{T\in p_S} \bar{v}(T)$. Let us define $\bar{\beta} = \{T \subset N : T \in p_S$ for some $S \in \beta\}$ and $\bar{\lambda}_T = \sum_{\substack{S\in\beta^*:\\ T\in p_S}} \lambda_S$, for each $T \in \bar{\beta}$. Given $i \in N$,

$$\sum_{\substack{T\in\bar{\beta}:\\ i\in T}} \bar{\lambda}_T = \sum_{\substack{T\in\bar{\beta}:\\ i\in T}} \left(\sum_{\substack{S\in\beta^*:\\ T\in p_S}} \lambda_S \right) = \sum_{\substack{S\in\beta^*:\\ i\in S}} \left(\sum_{\substack{T\in p_S:\\ i\in T}} \lambda_S \right) = \sum_{\substack{S\in\beta^*:\\ i\in S}} \lambda_S = 1.$$

Then, $\bar{\beta}$ is a balanced family with balancing weight vector $\left(\bar{\lambda}_T\right)_{T\in\bar{\beta}}$ and $\bar{\beta} \subset \pi$. Therefore,

$$v^*(N) = \sum_{S\in\beta^*} \lambda_S v(S) = \sum_{S\in\beta^*} \lambda_S \sum_{T\in p_S} \bar{v}(T) = \sum_{T\in\bar{\beta}} \left(\sum_{\substack{S\in\beta^*:\\ T\in p_S}} \lambda_S \right)$$
$$\bar{v}(T)) = \sum_{T\in\bar{\beta}} \bar{\lambda}_T \bar{v}(T) \leq v^\pi(N)$$

as desired.

Once we have presented the new feasibility notion, we can introduce the aspiration core definition.

The *aspiration core or balanced aspiration set* (Bennett 1983; see also Cross 1967; Albers 1979) is defined as,

$$AC(N, v) = \{x \in \mathbf{R}^n : x(N) \leq v^*(N) \quad \text{and} \quad x(S) \geq v(S)\ \forall\ S \subset N\}.$$

Bennett (1983) shows that $AC(N, v) \neq \emptyset$ for every game (N, v). Furthermore, the *aspiration core* can be seen as a natural non-empty extension of the core because when the latter is non-empty, both solutions coincide. For example, in the assignment game both solutions coincide.

The following lemma ensures that the *aspiration core* definition is consistent with the fact that only the basic coalitions play essential roles. The proof of this lemma uses standard techniques, and it is omitted.

Lemma 2 *If (N, v) is a game associated to $(N, \pi, \bar{v})$, then*
$AC(N, v) = \{x \in \mathbf{R}^n : x(N) \leq v^*(N)$ *and* $x(T) \geq \bar{v}(T)$ *for all* $T \in \pi\}$.

4 Main results

The following is one of the main results about replications of partitioning games.

Theorem (Kaneko and Wooders 1982) *Given (N, π), there exists an integer number m^0 such that for any positive integer number k and any $(N, v) \in GS(N, \pi)$, the replica games (N_r, v_r) have non-empty core, where $r = km^0$.*

This theorem states that there exists a subsequence of the generated sequence of replica games such that every game in the subsequence has a non-empty core. Theorem 1 below provides more information about the subsequence of non-empty cores and the relation of such subsequence with the *aspiration core* of the original game. We prove that over the subsequence of replica games where the non-emptiness of the core is guaranteed, cores of the replica games remain constant and they are equal to the aspiration core of the original game.

Theorem 1 *Given (N, π), there exists an integer number m^0 such that for any positive integer number k and any $(N, v) \in GS(N, \pi)$, $C(N_r, v_r) = AC(N, v)$ for all $r = km^0$.*

Proof First, we will prove that for all r, $C(N_r, v_r) \subset AC(N, v)$.

Let (N, v) be a game associated to $(N, \pi, \bar{v})$. Let x be in $C(N_r, v_r)$. Since $\bar{v}(T) = \bar{v}_r(T)$ for all $T \in \pi$, $x(T) \geq v(T) \geq \bar{v}(T)$ for all $T \in \pi$. Now, by Lemma 2, we only need to prove that $x(N) \leq v^*(N)$.

Let $\bar{x} = \Pi_{i=1}^{r} x$. As $x \in C(N_r, v_r)$, $\bar{x}(N_r) \leq v_r(N)$. So there exists a π_r−partition of N_r, p_{N_r}, such that

$$\bar{x}(T) \leq v_r(T) \quad \text{for all} \quad T \in p_{N_r}. \tag{3}$$

Let $\beta^* = \{T^* \in \pi : \varrho(T^*) = \varrho(T) \text{ for some } T \in p_{N_r}\}$. We will prove that β^* is a balanced family. Given $T^* \in \beta^*$, let h_{T^*} be defined by $h_{T^*} = \left|\{T \in p_{N_r} : \varrho(T) = \varrho(T^*)\}\right|$. Let us define $\lambda_{T^*} = \frac{h_{T^*}}{r}$. As p_{N_r} is a π_r−partition, each $T \in p_{N_r}$ has at most one member of each type (since $\varrho(T_j) = \varrho(T^*)$ with $T^* \in \pi$) and each pair (i, q) is in one and only one element of p_{N_r}. Then,

$$\sum_{\substack{T \in p_{N_r}: \\ (i,q) \in T \\ for\ some\ q}} 1 = r \quad \text{for all} \quad i \in N.$$

Now, given $i \in N$, we have that,

$$\begin{aligned} \sum_{\substack{T^* \in \beta^*: \\ i \in T^*}} \lambda_{T^*} &= \sum_{\substack{T^* \in \beta^*: \\ i \in T^*}} \frac{h_{T^*}}{r} = \frac{1}{r} \sum_{\substack{T^* \in \beta^*: \\ i \in T^*}} h_{T^*} = \frac{1}{r} \sum_{\substack{T^* \in \beta^*: \\ i \in T^*}} \left(\sum_{\substack{T \in p_{N_r}: \\ \varrho(T) = \varrho(T^*)}} 1 \right) \\ &= \frac{1}{r} \sum_{\substack{T \in p_{N_r}: \\ (i,q) \in T \\ for\ some\ q}} 1 = \frac{1}{r} r = 1. \end{aligned}$$

Therefore, β^* is a balanced family of N with balancing weights $(\lambda_{T_j^*})_{T_j^* \in \beta}$.

By definition of v_r and (3), we have that $x(T^*) \leq v(T^*)$ for all $T^* \in \beta^*$. Then,

$$x(N) = \sum_{\substack{T^* \in \beta^*: \\ i \in T^*}} \lambda_{T^*} x(T^*) \leq \sum_{\substack{T^* \in \beta^*: \\ i \in T^*}} \lambda_{T^*} v(T^*) \leq v^*(N). \tag{4}$$

So, we have proven that $C(N_r, v_r) \subset AC(N, v)$ for all r.

The other inclusion follows from the proof of Theorem 3.2 in Kaneko and Wooders (1982) and therefore it is omitted. □

The following limit theorem is the main result of this paper. It states that the collection of ϵ-*approximate cores* converges to the *aspiration core* when ϵ tends to zero.[10]

Theorem 2 *For every game* (N, v) *in* $GS(N)$,

$$\lim_{\epsilon \to 0} ApC_\epsilon(N, v) = AC(N, v).$$

Before proving Theorem 2, we need to establish some lemmas.

Remark 4 Note that Theorem 2 applies to every pair (N, π) and is independent of the original payoff function $\bar{v}$. Assume that we have a superadditive game (N, v). By Remark 1, we know that (N, v) is a partitioning game, and that there exists a pair (N, π) that generates (N, v). We can replicate (N, v) from (N, π), the numbers m^0 and r^* that appear in Theorem 1, and Kaneko and Wooders (1982) theorems will be depending on the choice of π. However, it is important to highlight that Theorem 2 is independent of the choice of π, and this is true for every π that we consider to generate (N, v). Therefore, Theorem 2 can be presented on superadditive games without reference to the original problem of partial cooperation.

Lemma 3 *Let* (N_r, v_r) *be the* $r-$*th replication of the game* (N, v), *then* $\frac{v_r(N)}{r} \leq v^*(N)$ *for all* r.

Proof Let p_{N_r} be a π_r-partition of N_r such that $v_r(N) = \sum_{T \in p_{N_r}} v_r(T)$. Let $\beta^* = \{T^* \in \pi : \varrho(T^*) = \varrho(T)$ for some $T \in p_{N_r}\}$. By the proof of Theorem 1, β^* is a balanced family of N with balancing weights $\lambda_{T^*} = \frac{h_{T^*}}{r}$ where $h_{T^*} = \left|\{T \in p_{N_r} : \varrho(T) = \varrho(T^*)\}\right|$, for all $T^* \in \beta^*$. Therefore,

$$\frac{v_r(N)}{r} = \sum_{T \in p_{N_r}} \frac{v_r(T)}{r} = \sum_{T^* \in \beta^*} h_{T^*} \frac{v(T^*)}{r} = \sum_{T^* \in \beta^*} \lambda_{T^*} v(T^*) \leq v^*(N).$$

□

[10] The limit notion is the classical one used in set theory. Given a set X and an indexed collection of subsets $(A_\epsilon)_{\epsilon \in (0,\infty)}$ of X such that $A_\epsilon \subset A_{\epsilon'}$ if $\epsilon < \epsilon'$, *the limit of* A_ϵ *when* ϵ *tends to zero* is,

$$\lim_{\epsilon \to 0} A_\epsilon = \bigcap_{\epsilon > 0} A_\epsilon.$$

In a similar way to the ϵ-core definition, we define the *ϵ-aspiration core* of a game (N, v) as the set,

$$AC_\epsilon(N, v) = \{x \in \mathbf{R}^n : x(N) \le v^*(N) \text{ and } x(S) \ge v(S) - \epsilon\,|S| \text{ for all } S \subset N\}.$$

The next lemma presents the relation between the ϵ-core of a replicated game and the ϵ-aspiration core of the original game.

Lemma 4 *For all game (N, v) in $GS(N)$, all $\epsilon > 0$ and all integer r,*

$$EC_\epsilon(N_r, v_r) = AC_\epsilon(N, v) \cap \{x \in \mathbf{R}^n : rx(N) \le v_r(N_r)\}.$$

Proof Since (N, v) is in $GS(N)$, there exists $(N, \pi, \bar{v})$ such that (N, v) is associated to $(N, \pi, \bar{v})$. Suppose x is in $EC_\epsilon(N_r, v_r)$. Then,

$$rx(N) \le v_r(N_r). \tag{5}$$

We will prove that $x \in AC_\epsilon(N, v)$. By Lemma 3 and (5),

$$x(N) \le v^*(N). \tag{6}$$

As $x \in EC_\epsilon(N_r, v_r)$, then

$$x(T) \ge v(T) - \epsilon\,|T| \quad \text{for all} \quad T \in \pi.$$

Given $S \subset N$, let $p_S \in \mathcal{P}^\pi(S)$ such that, $v(S) = \sum_{T \in p_S} \bar{v}(T)$. Then,

$$\begin{aligned} v(S) &= \sum_{T \in p_S} \bar{v}(T) \le \sum_{T \in p_S} v(T) \le \sum_{T \in p_S} x(T) + \epsilon\,|T| = \sum_{T \in p_S} x(T) + \sum_{T \in p_S} \epsilon\,|T| \\ &= x(S) + \epsilon\,|S|. \end{aligned} \tag{7}$$

Therefore, by (6) and (7), $x \in AC_\epsilon(N, v)$.
To see the other inclusion, let x be such that $rx(N) \le v_r(N_r)$. If $\bar{x} = \Pi_{i=1}^r x$, then

$$\bar{x}(N_r) \le v_r(N_r). \tag{8}$$

Now, let us suppose x is in $AC_\epsilon(N, v)$. If $\bar{x} = \Pi_{i=1}^r x$, then $\bar{x}(T) = x(T) \ge v(T) - \epsilon\,|T|$ for all $T \in \pi$. By definition of (N_r, v_r),

$$\bar{x}(T) \ge v_r(T) - \epsilon\,|T| \quad \text{for all} \quad T \in \pi_r. \tag{9}$$

Therefore, by (8), (9) and Lemma 2.1 of Kaneko and Wooders (1982), $x \in EC_\epsilon(N_r, v_r)$. □

Remark 5 From the previous lemma, the following three statements hold:

(i) $EC_\epsilon(N_r, v_r)$ is always included in $AC_\epsilon(N, v)$.
(ii) $AC_\epsilon(N, v)$ is always non-empty but in cases in which $\frac{v_r(N_r)}{r} < v^*(N)$, we have that $EC_\epsilon(N_r, v_r)$ is empty.
(iii) By Theorem 1, there is an integer number m^0 such that for any positive integer number k, $\frac{v_r(N_r)}{r} = v^*(N) - \epsilon\,|N|$ for all $r = km^0$. Then, $EC_\epsilon(N_r, v_r)$ remains constant satisfying $EC_\epsilon(N_r, v_r) = AC_\epsilon(N, v)$ for all $r = km^0$, i.e., over a subsequence of replica games, the payoffs with the equal-treatment property in the ϵ-cores of the replica games remain constant, and they are equal to the ϵ-aspiration core of the original game.

Now, we are ready to prove Theorem 2.

Proof of Theorem 2. By Lemma 4, $EC_\epsilon(N_r, v_r) \subset AC_\epsilon(N, v)$ for all integer r, for all $\epsilon > 0$. Then, $ApC_\epsilon(N, v) \subset AC_\epsilon(N, v)$ for all $\epsilon > 0$. By Remark 5 (iii), there exists an integer number m^0 such that for all positive integer number k, $EC_\epsilon(N_r, v_r) = AC_\epsilon(N, v)$ for all $r = km^0$. Therefore,

$$ApC_\epsilon(N, v) = AC_\epsilon(N, v) \quad \text{for all} \quad \epsilon > 0.$$

As $ApC_{\epsilon_1}(N, v) \subset ApC_{\epsilon_2}(N, v)$ if $\epsilon_1 \leq \epsilon_2$, we have

$$\lim_{\epsilon \to 0} ApC_\epsilon(N, v) = \bigcap_{\epsilon > 0} ApC_\epsilon(N, v) = \bigcap_{\epsilon > 0} AC_\epsilon(N, v) = AC(N, v).$$

□

Remark 6 We have proven that

$$ApC_\epsilon(N, v) = AC_\epsilon(N, v) \quad \text{for all} \quad \epsilon > 0.$$

So that, the ϵ-approximate core and the ϵ -aspiration core coincide for all $\epsilon > 0$.

The next last theorem shows the "closedness" between ϵ-cores of replicated games and the ϵ-aspiration core of the original games if the number of replications is large enough.

Theorem 3 *For any* $(N, v) \in GS(N, \pi)$ *and any* $\epsilon > 0$, *there is an integer* r^* *such that for all* $r \geq r^*$,

$$\begin{aligned} &EC_\epsilon(N_r, v_r) \subset AC_\epsilon(N, v) \text{ and} \\ &AC_\epsilon(N, v) \backslash EC_\epsilon(N_r, v_r) \subset \{x \in \mathbf{R}^n : v^*(N) - \epsilon < x(N) \leq v^*(N)\}. \end{aligned} \tag{10}$$

Proof The first inclusion follows from Lemma 4. Let us see the second inclusion. Given $\epsilon > 0$, by the proof of Theorem 3.4 in Kaneko and Wooders (1982), there exists r^* such that [11]

$$rv^*(N) - v_r(N_r) < r\epsilon \quad \text{for all} \quad r \geq r^*. \tag{11}$$

Now, if $r \geq r^*$ and $x \in AC_\epsilon(N, v) \backslash EC_\epsilon(N_r, v_r)$, then $v_r(N_r) < rx(N) \leq rv^*(N)$ (see Lemma 4). Then, by (11), $v^*(N) - \epsilon < x(N) \leq v^*(N)$. □

Let us see some implications of Theorem 3. First, for small $\epsilon > 0$, the ϵ-aspiration core of a (original) game is very "closed" to the set of payoffs, with the equal-treatment property, in the ϵ-core of (N_r, v_r) when r is large enough.

Second, it is clear that,[12]

$$AC(N, v) - \{(\epsilon, \epsilon, \ldots, \epsilon)\} \subset AC_\epsilon(N, v) \cap \{x \in \mathbf{R}^n : x(N) \leq v^*(N) - \epsilon\}.$$

So, by (10), the payoffs in the *aspiration core* of the (original) game "minus" ϵ are in the ϵ-core of the r -th replicate game if r is large enough.

Because of previous results, we could say that the ideas of *approximate core* and *aspiration core* are in complete accordance. Nevertheless, the *aspiration core* notion has the advantage that it does not need to introduce the notions of replicated games and ϵ-cores; it only needs to calculate the set of feasible payoffs that is obtained when the players are organized by balanced families, as well as to select those payoffs that are not blocked by the basic coalitions.

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[11] See footnote 7.

[12] If $A \subset \mathbf{R}^n$, then an element x is in $A - \{(\epsilon, \ldots, \epsilon)\}$ if and only if $x = (y_1 - \epsilon, \ldots, y_n - \epsilon)$ with $y \in A$.

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